II YEAR - III SEMESTER
COURSE CODE: 7BMAA3

ALLIED COURSE - III – ANCILLARY MATHEMATICS III

Unit – I

Partial Differential Equations – Formation of Partial Differential Equations by eliminating arbitrary constants and arbitrary functions – Complete, Particular, Singular and General integral.

Partial Differential Equations

Def: (Differential equation).

An equation involving derivatives (or) differentials of one or more dependent variables with respect to one or more independent variables is called differential equation.

Equation:

A mathematical statement that two things are equal. It consist of two expressions, one on each side of an 'equals' sign.

Eg. 1. $12 = 7+5$ 2. $x = 7+5$.

$x^2 - 7x + 5 \rightarrow$ is not an equation, but its is an expression.

Classification of equations:

1. polynomial (or) algebraic equation.
2. Parametric equation
3. Diophantine equation
4. transcendental equation

so on.

Note:

1. In general an equality is an equation.
2. '=' invented by Robert Recorde.

ODE :

A differential Equation involving derivatives with respect to a single independent variable.

Eg: $\dfrac{dy}{dx} = x + \sin x$

here $x \rightarrow$ independent & $y \rightarrow$ dependent.

PDE :

A differential equation involving partial derivatives with respect to ~~one or~~ more than one independent variable is called P.D.E.

Eg. 1. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

here the eqn depends on three independent variables x, y & z.

2. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

Constants/co-efficients.

Conventions :

We consider x and y as independent variables and z as dependent variable. And

$p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$

and $t = \dfrac{\partial^2 z}{\partial y^2}$.

Formation of partial differential equation:

A PDE can be derived in two ways.

I. By eliminating arbitrary constants

II. By eliminating arbitrary functions.

I. Arbitrary constants elimination method:

Let $f(x, y, z, a, b) = 0$ be a relation between x, y, z involving two arbitrary constants a and b.

In order to eliminate a & b, we differentiate ① with respect to x and y respectively and express the equation using p, q, r, s and t.

Eg.

①. Form partial differential equation by eliminating the arbitrary constants a and b from $z = (x+a)^2 + (y+b)^2 + c^2$.

Soln: $z = (x+a)^2 + (y+b)^2 + c^2 \rightarrow ①$

Differentiating partially ① with respect to x and y

we get $\frac{\partial z}{\partial x} = 2(x+a)(1) + 0 + 0 = 2(x+a)$

$\Rightarrow p = 2(x+a) \Rightarrow x+a = p/2$.

and $\frac{\partial z}{\partial y} = 0 + 2(y+b)(1) + 0 = 2(y+b)$

$q = 2(y+b) \Rightarrow (y+b) = q/2$

Substitute $(x+a)$ and $(y+b)$ in ①, we get.

$z = (p/2)^2 + (q/2)^2 + c^2 = \frac{p^2}{4} + \frac{q^2}{4} + c^2$

$\Rightarrow \boxed{4z = p^2 + q^2 + 4c^2}$, the required equation.

(2) Form differential equation by eliminating the arbitrary constants a and b ~~form~~ from $z = axy + b$.

Soln. $z = axy + b \rightarrow (1)$

Differentiating (1) partially with respect to x and y

$$\frac{\partial z}{\partial x} = \cancel{ay+} \; ay(1) + 0 = p = ay.$$

$$\Rightarrow p/y = a \quad \text{ie} \quad \boxed{a = p/y} \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = ax(1) + 0, \Rightarrow q = ax$$

$$\Rightarrow \boxed{a = q/x} \rightarrow (3).$$

$$(2) = (3) \quad q/x = a = p/y \quad \Rightarrow p/y = q/x$$

$$\Rightarrow p/y - q/x = 0$$

$\therefore$ The required eqn is $= \boxed{px - qy = 0}$

(3). $z = axe^{y} + \frac{1}{2}a^{2}e^{2y} + b$

Soln. $z = axe^{y} + \frac{1}{2}a^{2}e^{2y} + b \rightarrow (1)$

$$\frac{\partial z}{\partial x} = ae^{y}(1) + 0 + 0$$

$$\boxed{p = \frac{\partial z}{\partial x} = ae^{y}} \rightarrow (2) \qquad \because \frac{\partial z}{\partial x} = p$$

$$\frac{\partial z}{\partial y} = axe^{y}(1) + \frac{1}{2}a^{2}e^{2y}\,2(1) + 0$$

$$= axe^{y} + \frac{1}{\cancel{2}}a^{2}e^{2y}\cdot\cancel{2} = \boxed{axe^{y} + a^{2}e^{2y} = q}$$

$$\because \frac{\partial z}{\partial y} = q$$

$$q = axe^{y} + a^{2}e^{2y} \longrightarrow (3)$$

From (2) we get $a = \frac{p}{e^y} = pe^{-y} \Rightarrow \boxed{a = pe^{-y}} \rightarrow (4)$

Take (4) in (3), we get

$$q = p\cancel{e^{-y}}\,x\,\cancel{e^{y}} + (pe^{-y})^2 e^{2y}$$

$$= px + p^2 \cancel{e^{-2y}}\,\cancel{e^{2y}}$$

$$= px + p^2$$

$\boxed{q = p(x+p)}$ the required equation.

(4). Form PDE by eliminating arbitrary constants a, b, c from (i) $z = (x+a)(y+b)$ (ii) $z = ax+by+ab$.

Soln. (i) ~~Let~~ $z = (x+a)(y+b) \longrightarrow (1)$

$$\frac{\partial z}{\partial x} = (y+b)(1+0) = (y+b)$$

$\boxed{p = (y+b)} \rightarrow (2)$. $\left[\because \frac{\partial z}{\partial x} = p\right]$

Differentiating partially (1) with respect to y.

$$\frac{\partial z}{\partial y} = (x+a)(1+0) = x+a$$

$\boxed{q = x+a} \longrightarrow (3)$ $\left[\because \frac{\partial z}{\partial y} = q\right]$

Substituting (2) & (3) in (1), we get

$z = qp \Rightarrow \boxed{z = pq}$

The required equation.

Function elimination method:

① Form a P.D.E by eliminating the arbitrary function f from $z = f(y/x)$.

Soln $z = f(y/x) \rightarrow ①$

Differentiating ① partially with respect to x, we get

$$\frac{\partial z}{\partial x} = f'(y/x)\left(\frac{x(0) - y(1)}{x^2}\right)$$

$$p = f'(y/x)(-y/x^2) \rightarrow ②$$

Differentiating ① partially with respect to y, we get

$$\frac{\partial z}{\partial y} = q = f'(y/x)\left(\frac{x(1) - y(0)}{x^2}\right)$$

$$q = f'(y/x)(1/x)$$

$$qx = f'(y/x) \rightarrow ③$$

take ③ in ②, we get

$$p = q\cancel{x}(-y/x^{\cancel{2}})$$

$$p = q(-y/x) \Rightarrow px = -qy.$$

$\therefore$ The required eqn is $= \boxed{px + qy = 0}$

② Form a PDE by eliminating the arbitrary functions f and g from $z = f(2x+y) + g(3x-y)$.

Soln. $z = f(2x+y) + g(3x-y) \rightarrow ①$

Differentiating ① partially w.r.to x, we get

$$\frac{\partial z}{\partial x} = f'(2x+y)\,2(1) + g'(3x-y)(3)$$

$$\Rightarrow p = 2f'(2x+y) + 3g'(3x-y) \rightarrow ②$$

Illy. Differentiating with respect to y, we get

$$\frac{\partial z}{\partial y} = f'(2x+y)(1) + g'(3x-y)(-1)$$

ie $$q = f'(2x+y) - g'(3x-y) \rightarrow (3)$$

Again differentiating (2) partially w.r.to x, we get

$$\frac{\partial^2 z}{\partial x^2} = 2f''(2x+y)(2) + 3g''(3x-y)(3)$$

$$\Rightarrow r = 4f''(2x+y) + 9g''(3x-y) \rightarrow (4)$$

Illy (3) will become

$$\frac{\partial^2 z}{\partial y^2} = f''(2x+y)(1) - g''(3x-y)(-1)$$

$$\Rightarrow t = f''(2x+y) + g''(3x-y) \rightarrow (5)$$

(5) $\times 4$ $-$ (4) $\Rightarrow$

$$4t = 4f''(2x+y) + 4g''(3x-y)$$

$$\underset{(-)}{}r = \underset{(-)}{}4f''(2x+y) \underset{(-)}{+} 9g''(3x-y)$$

$$4t - r = -5g''(3x-y)$$

$$\Rightarrow \frac{4t-r}{-5} = g''(3x-y) \rightarrow (6)$$

(5) $\times 9$ $-$ (4) is

$$9t = 9f''(2x+y) + 9g''(3x-y)$$

$$\underset{(-)}{}r = \underset{(-)}{}4f''(2x+y) \underset{(-)}{+} 9g''(3x-y)$$

$$9t - r = 5f''(2x+y)$$

$$\Rightarrow \frac{9t-r}{5} = f''(2x+y) \rightarrow (7)$$

Now differentiate (2) partially with respect to y, we get

$$\frac{\partial^2 z}{\partial x \partial y} = 2f''(2x+y)(1) + 3g''(3x-y)(-1)$$

$$\Rightarrow s = 2f''(2x+y) - 3g''(3x-y) \rightarrow (8)$$

take (6) & (7) in 8, we get.

$$s = 2\left(\frac{9t-r}{5}\right) - 3\left(\frac{4t-r}{-5}\right)$$

$$5s = 18t - 2r + 12t - 3r$$

$$5s = 30t - 5r$$

$$s = 6t - r \Rightarrow \boxed{s + r = 6t}$$, the required.

(3) Form PDE by eliminating the function f from $z = f(x+ay)$.

Soln. $z = f(x+ay) \rightarrow (1)$

$$\frac{\partial z}{\partial x} = f'(x+ay)(1)$$

$$\Rightarrow p = f'(x+ay) \rightarrow (2)$$

then $$\frac{\partial z}{\partial y} = q = f'(x+ay)(0 + a(1))$$

$$q = af''(x+ay) \rightarrow (3).$$

Substitute (2) in (3), we get

$$q = ap \Rightarrow \boxed{q - ap = 0}$$

Classification of integrals:

We have seen that the partial differential equation of the form $f(x,y,z,p,q)=0 \rightarrow (1)$ can be derived from $\phi(x,y,z,a,b)=0 \rightarrow (2)$ by eliminating the constants a, b.

Hence (2) is ~~the~~ a solution of (1).

C.I

The solution of (1) which as many arbitrary constants as there are independent variables is called the complete integral of (1).

Particular Integral:

P.I of (1) is obtained by giving particular values of arbitrary constants.

Singular Integral:

Now consider the complete solution $\phi(x,y,z,a,b)=0$. This can be thought of as a two parameter family of surfaces. The envelope of the family of surfaces is obtained by eliminating a & b from the three equations.

$$\phi(x,y,z,a,b)=0, \quad \frac{d\phi}{da}=0, \quad \frac{d\phi}{db}=0.$$

The resulting relation between x, y & z is called the singular integral of (1).

Note: This can not be obtained from the complete integral by giving particular values to the arbitrary constants.

General integral:

In the complete integral $\phi(x,y,z,a,b)=0$, put $b=f(a)$. Then it becomes $\phi(x,y,z,a,f(a))=0$ which represents a one parameter of family of surfaces.

Its envelope is obtained by eliminating a from the two equations $\phi(x,y,z,a,f(a))=0$; $\frac{d\phi}{da}=0$.

The resulting relation between x, y and z is called the general integral of ①.

Unit – II

Solving Lagrange's linear equation $Pp + Qq = R$ .Solution of equations of Standard types $f(p, q) = 0$, $z = px + qy + f(p, q)$, $f(z, p, q) = 0$, $f_1(x, p) = f_2(y, q)$.

UNIT - II

METHOD OF SOLVING FIRST ORDER PDE

A PDE which involves only the first order partial derivatives p and q, is called a I order partial differential equation.

* The general form of I order PDE is $f(x,y,z,p,q)=0$.
* A PDE which is linear in p and q is of the form
  $Pp+Qq=R$, where P, Q, R are functions of x, y, z.

  Eg $yzp+zxq=xy$, where $P=yz$, $Q=zx$, $R=xy$.

Lagrange's Equation:

A PDE which is linear in p and q is of the form

$Pp+Qq=R \rightarrow (1)$, where $P, Q, \& R$ are functions of $x, y \& z$, is called Lagrange's Linear equation.

Auxiliary Equation:

The system of equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \rightarrow (2)$ is called the auxiliary equation of (1). If $u=a$, $v=b$ are two solutions of (2), then $\phi(u,v)=0$ is the general solution of (1). Auxiliary equation can be solved by

1). Method of Grouping 2). Method of multiplication.

Method of Grouping:

In the auxiliary equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, if the variables can be separated in any pairs of equations, then we get a solution of the form $u(x,y)=a$ and $v(x,y)=b$.

II Method of multipliers:

Choose any three multipliers l, m, n which may be constants (or) functions of x, y, z. We have.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such ~~at~~ that $lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$.

If $l\,dx + m\,dy + n\,dz$ is an exact differential then on Integration we get a solution $u = a$.

Note:

The multipliers l, m, n are called Lagrange's multipliers.

Working rule to solve the Lagrange's linear equation $Pp + Qq = R$:

(i) Form the auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

(ii) Solve the auxiliary equation and get two independent solutions $u = a$ & $v = b$.

(iii) The general solution is $\phi(u, v) = 0$.

Eg ①. Solve $2p + 3q = 1$.

Soln.

This is of the form $Pp + Qq = R \rightarrow (1)$

W.K.T the auxiliary equation of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\therefore$ the auxiliary equation is $\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1} \rightarrow (2)$

taking the first two ratios of (2)

$$\frac{dx}{2} = \frac{dy}{3} \Rightarrow 3\,dx = 2\,dy$$

$$\Rightarrow 3\,dx - 2\,dy = 0$$

On integrating both sides, we get

$$3\int dx - 2\int dy = \int 0$$

$$3x - 2y = a. \quad (\because a \text{ is a constant}).$$

taking the last two ratio's, we get.

$$\frac{dy}{3} = \frac{dz}{1} \Rightarrow dy = 3dz$$

$$dy - 3dz = 0$$

On integrating both sides, we get

$$\int dy - 3\int dz = \int 0$$

$$y - 3z = b \quad (\because b \text{ is a constant}).$$

$\therefore$ the general solution ~~is~~ $\phi(u, v) = 0$ is

$$\phi(3x-2y,\ y-3z) = 0. \quad [\because u=a, v=b]$$

(2). Find the general solution of $zp + x = 0$.

Soln

Given eqn $zp + x = 0$.

ie $zp = -x \Rightarrow zp + 0q = -x$.

This is of the form $Pp + Qq = R \rightarrow (1)$.

ie $P = z$, $Q = 0$, $R = -x$.

The auxiliary equation of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \text{ is}$$

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \rightarrow (2)$$

Taking the first and last ratio's of (2)

$$\frac{dx}{z} = \frac{dz}{-x}$$

$$\Rightarrow -x\,dx = z\,dz$$

$$\Rightarrow z\,dz + x\,dx = 0$$

On integrating both sides, we get

$$\int z\,dz + \int x\,dx = \int 0$$

$$\frac{z^2}{2} + \frac{x^2}{2} = c$$

$$z^2 + x^2 = 2c$$

$$z^2 + x^2 = a \qquad (\because a = 2c)$$

taking the second ratio,

$$\frac{dy}{0} \Rightarrow dy = 0.$$

On integrating, we get

$$\int dy = \int 0$$

$$y = b \qquad [\because b \text{ is a constant}]$$

$\therefore$ The general solution $\phi(u, v) = 0$ is

$$\phi(z^2 + x^2, y) = 0.$$

—— x ——

H.W

(1) Solve $x^2 p + y^2 q = z^2$

Ans: $\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0.$

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H.W

3. Solve $x^2p + y^2q = z^2$.

Soln Given equation $x^2p+y^2q = z^2 \rightarrow (1)$

This is of the form $Pp + Qq = R$

& The auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

∴ The auxiliary equation of (1) is

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \rightarrow (2).$$

Take first two ratio's of (2)

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \cancel{y^2dx = x^2dy}\ x^{-2}dx = y^{-2}dy$$

$$\cancel{y^2dx - x^2dy} = 0$$

$$\Rightarrow x^{-2}dx - y^{-2}dy = 0$$

Integrating on both sides, we get

$$\int x^{-2}dx - \int y^{-2}dy = \int 0$$

$$\frac{x^{-2+1}}{-2+1} - \frac{y^{-2+1}}{-2+1} = a$$

$$\frac{x^{-1}}{-1} - \frac{y^{-1}}{-1} = a$$

$$-\frac{1}{x} + \frac{1}{y} = a$$

$$\boxed{\frac{1}{y} - \frac{1}{x} = a}$$

Taking last two ratios of (2)

$$\frac{dy}{y^2} = \frac{dz}{z^2} \Rightarrow y^{-2}dy = z^{-2}dz$$

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$\Rightarrow y^{-2} dy - z^{-2} dz = 0$

Integrating on both sides, we get

$$\int y^{-2} dy - \int z^{-2} dz = \int 0$$

$$\frac{y^{-2+1}}{-2+1} - \frac{z^{-2+1}}{-2+1} = b$$

$$\frac{y^{-1}}{-1} - \frac{z^{-1}}{-1} = b$$

$$-\frac{1}{y} + \frac{1}{z} = b$$

$$\boxed{\frac{1}{z} - \frac{1}{y} = b}$$

$\therefore$ The general solution $\phi(u, v) = 0$ is

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0.$$

④ Solve $p \cot x + q \cot y = \cot z$.

Soln

Given equation is $\cot x\, p + \cot y\, q = \cot z$.

A.E is $\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$

First two ratios

$$\frac{dx}{\cot x} = \frac{dy}{\cot y}$$

$$\Rightarrow \frac{dx}{\cot x} - \frac{dy}{\cot y} = 0.$$

Integrating bothsides

$$\int \frac{dx}{\cot x} - \int \frac{dy}{\cot y} = \int 0$$

$$\int \tan x\, dx - \int \tan y\, dy = \int 0 \quad \left[\because \frac{1}{\cot x} = \tan x\right]$$

$$\log \sec x - \log \sec y = \log a$$

$$\Rightarrow \log \frac{\sec x}{\sec y} = \log a$$

$$\frac{\sec x}{\sec y} = a$$

$$\frac{\frac{1}{\cos x}}{\frac{1}{\cos y}} = a \Rightarrow \boxed{\frac{\cos y}{\cos x} = a.}$$

Taking last two ratios of (2)

$$\frac{dy}{\cot y} = \frac{dz}{\cot z} \Rightarrow \frac{dy}{\cot y} - \frac{dz}{\cot z} = 0.$$

Integrating on both sides, we get

$$\int \frac{dy}{\cot y} - \int \frac{dz}{\cot z} = \int 0$$

$$\int \tan y\, dy - \int \tan z\, dz = \int 0$$

$$\log \sec y - \log \sec z = \log b$$

$$\log \frac{\sec y}{\sec z} = \log b$$

$$\frac{\sec y}{\sec z} = b$$

$$\frac{1/\cos y}{1/\cos z} = b \Rightarrow \boxed{\frac{\cos z}{\cos y} = b}$$

$\therefore$ The general solution $\phi(u,v)=0$ is

$$\phi\left(\frac{\cos y}{\cos x}, \frac{\cos z}{\cos y}\right) = 0.$$

5. Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

<u>Soln</u> Given $\sqrt{x}\,p + \sqrt{y}\,q = \sqrt{z} \rightarrow (1)$

A.E is $\frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \rightarrow (2)$

Taking the first two ratios of (2)

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

$$\Rightarrow \frac{dx}{\sqrt{x}} - \frac{dy}{\sqrt{y}} = 0.$$

Integrating on both sides, we get

$$\int \frac{dx}{\sqrt{x}} - \int \frac{dy}{\sqrt{y}} = \int 0$$

$$\cancel{2\sqrt{y} - 2\sqrt{z} = c}$$
$$\cancel{\sqrt{y} - \sqrt{z} = a} \quad [\because \cancel{\ddagger c = a}]$$

$$2\sqrt{x} - 2\sqrt{y} = c$$
$$\sqrt{x} - \sqrt{y} = 2c$$
$$\boxed{\sqrt{x} - \sqrt{y} = a} \quad (\because 2c = a)$$

Taking the last two ratio's of 2, we get

$$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \Rightarrow \frac{dy}{\sqrt{y}} - \frac{dz}{\sqrt{z}} = 0.$$

Integrating on both sides, we get

$$\int \frac{dy}{\sqrt{y}} - \int \frac{dz}{\sqrt{z}} = \int 0$$

$$2\sqrt{y} - 2\sqrt{z} = c$$
$$\boxed{\sqrt{y} - \sqrt{z} = b} \quad [\because b = 2c]$$

$\therefore$ The general solution $\phi(u,v)$ is

$$\phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0.$$

H.W

1. Solve $y^2zp + x^2zq = xy^2$

Ans: $\phi(x^3 - y^3, x^2 - z^2) = 0.$

8). Find the general solution of $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

Soln.

Auxiliary equation is

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \rightarrow (1)$$

taking x, y, z as Lagrangian multipliers, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2)} = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2y^2} - \cancel{x^2z^2} + \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{z^2x^2} - \cancel{z^2y^2}} = 0$$

$x\,dx + y\,dy + z\,dz = 0$

Integrating,

$$\int x\,dx + \int y\,dy + \int z\,dz = \int 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as Lagrangian multipliers in (1), we get

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\frac{\cancel{x}(y^2-z^2)}{\cancel{x}} + \frac{\cancel{y}(z^2-x^2)}{\cancel{y}} + \frac{\cancel{z}(x^2-y^2)}{\cancel{z}}} = 0.$$

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2-z^2+z^2-x^2+x^2-y^2} = 0$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = \int 0$

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$$xyz = b.$$

$\therefore$ the general solution $\phi(a,b)=0$ is

$$\phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ xyz\right) = 0.$$

(9). Solve. $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

Soln A.E is

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(-y+x)} \rightarrow (1)$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as Lagrangian multipliers in (1), we get

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\frac{x^{\cancel{2}}(y-z)}{\cancel{x}} + \frac{y^{\cancel{2}}(z-x)}{\cancel{y}} + \frac{z^{\cancel{2}}(x-y)}{\cancel{z}}} = 0$$

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\cancel{xy} - \cancel{zx} + \cancel{yz} - \cancel{xy} + \cancel{zx} - \cancel{yz}} = 0$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

Integrating $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = \int 0$

$$\log x + \log y + \log z = \log a$$

$$\log xyz = \log a \Rightarrow \boxed{xyz = a}$$

Taking the Lagrangian multipliers as $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$,

we get

$$\frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{\frac{\cancel{x^2}(y-z)}{\cancel{x^2}} + \frac{\cancel{y^2}(z-x)}{\cancel{y^2}} + \frac{\cancel{z^2}(x-y)}{\cancel{z^2}}} = 0$$

$$\frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{\cancel{y} - \cancel{z} - \cancel{x} + \cancel{z} + \cancel{x} - \cancel{y}} = 0$$

Integrating, we get

$$\int \frac{dx}{x^2} + \int \frac{dy}{y^2} + \int \frac{dz}{z^2} = \int 0$$

$$\int x^{-2}dx + \int y^{-2}dy + \int z^{-2}dz = \int 0$$

$$\frac{x^{-2+1}}{-2+1} + \frac{y^{-2+1}}{-2+1} + \frac{z^{-2+1}}{-2+1} = b$$

$$\frac{x^{-1}}{-1} + \frac{y^{-1}}{-1} + \frac{z^{-1}}{-1} = b$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = b.$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -b = b' \qquad [\because b' = -b]$$

$\therefore$ The general solution $\phi(c, b') = 0$ is $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) =$

UNIT – II → II PART.

Some standard forms:

TYPE 1. Equations of the form $f(p,q)=0$.

Consider an equation of the form $f(p,q)=0 \rightarrow (1)$

clearly $z = ax + by + c$, where a & b are such that $f(a,b)=0$ satisfies (1).

Now, solving for b from $f(p,q)=0$, we get $b = g(a)$.

The complete integral is $z = ax + y\,g(a) + c \rightarrow (2)$

To find the general solution put $c = \phi(a)$ in (2)

We get $z = ax + y\,g(a) + \phi(a) \rightarrow (3)$

Differentiating (3) w.r.to a we get

$$x + y\,g'(a) + \phi'(a) = 0 \rightarrow (4)$$

Eliminating a from (3) & (4) we get the general solution.

To find the singular integral, we have to eliminate a and c from the following three equations.

$$z = ax + g(a)y + c; \quad 0 = x + yg'(a); \quad 0 = 1.$$

The last equation assumes that there is no singular integral in this case.

SOLVED PROBLEMS:

① . Solve $pq + p + q = 0$.

Soln. Complete integral:

Given eqn. $pq + p + q = 0 \rightarrow ①$

This is of the form $f(p, q) = 0$ & its solution is $z = ax + by + c \rightarrow ②$

where $ab + a + b = 0$ is a solution of ①

Now $ab + a + b = 0$

$ab + b + a = 0$

$b(a+1) + a = 0$

$$\Rightarrow \boxed{b = \frac{-a}{a+1}} \rightarrow ③$$

Take ③ in ②, we have

$$z = ax + y\left(\frac{-a}{a+1}\right) + c \rightarrow ④$$ is the complete integral.

General integral:

put $c = \phi(a)$ in ④

$$z = ax - \left(\frac{a}{a+1}\right)y + \phi(a) \rightarrow ⑤.$$

Differentiating ⑤ w.r.t a, we get

$$0 = x - \frac{(a+1)(1) - a(1)}{(a+1)^2}\, y + \phi'(a) \rightarrow ⑥.$$

$$0 = x - \frac{1}{(1+a)^2} y + \varphi'(a) \longrightarrow (6)$$

Eliminate a from (5) & (6), we get general solution.

<u>Singular integral:</u>

Differentiating (4) w.r.t a, we get

$$0 = x - \frac{1}{(a+1)^2} y + 0$$

Differentiating (4) w.r.t k, we get

$$0 = 0 + 0 + 1$$

$0 = 1$. which is a contradiction.

$\therefore$ There is no singular integral.

TYPE 2 :

Equation of the form $z = px + qy + f(p,q)$.

The complete solution is given by $z = ax + by + f(a,b)$

The general & singular integrals are obtained by usual method

Problem.

1. Solve $z = px + qy + (q/p) - p$

Soln The complete integral is given by $z = ax + by + (b/a) - a$ → ①

To find the singular integral, we differentiate ①

w.r.t a and b, we get

$$0 = x + \frac{a(0) - b(1)}{a^2} - 1$$

$$0 = x - \frac{b}{a^2} - 1 \rightarrow ②$$

and

$$0 = 0 + y + \frac{a(1) - b(0)}{a^2} - 0$$

$$= y + \frac{a}{a^2}$$

$$\frac{1}{a} = -y \Rightarrow \boxed{a = -1/y} \text{ and}$$

$$② \Rightarrow \quad 0 = x - \frac{b}{(-1/y)^2} - 1$$

$$= x - y^2 b - 1$$

$$= -y^2 b = -x + 1$$

$$b = \frac{-x+1}{-y^2}$$

$$\boxed{b = \frac{x-1}{y^2}}$$

Substituting the value of a and b in ①, we get

$$z = \left(-1/y\right)x + \left(\frac{x-1}{y^2}\right)y + \frac{\frac{x-1}{y^2}}{-1/y} - \left(-1/y\right)$$

$$z = -\frac{x}{y} + \frac{x-1}{y} - \frac{x-1}{y} + \frac{1}{y}$$

$$zy = -\cancel{x} + \cancel{x} - \cancel{1} - x + \cancel{1} + 1$$

$$\boxed{zy = 1 - x}$$

Hence the singular solution is $yz = 1 - x$.

General solution:

put $b = \phi(a)$ in ①, we get

$$z = ax + \phi(a)y + \frac{\phi(a)}{a} - a \rightarrow ③$$

Differentiating ③ w.r.t. a, we get

$$0 = x + \phi'(a)y + \frac{a\phi'(a) - \phi(a)(1)}{a^2} - 1$$

$$= x + \phi'(a)y + \frac{\phi(a)}{a} - \frac{\phi(a)}{a^2} - 1 \rightarrow ④$$

Eliminating a from ③ & ④ we get the general solution.

Unit – III

Laplace Transform – Definition – Laplace transform of some Standard Functions – problems – Inverse Laplace Transform – Standard formulae – problems.

Laplace Transform :-

⇒ It's named in honour of great French Mathematician, Pierre Simon De Laplace (1749-1827)

⇒ The best way to convert differential equations into algebraic equations is use of Laplace Transformation.

Definition

*. Let $f(x)$ be a function defined on $[0, \infty]$

The Laplace Transform L is defined by

$$\boxed{L[f(x)] = \int_0^\infty e^{-sx} f(x)\, dx = F(s)}$$

*. The above Laplace Tranform L acts on any function $f(x)$ for which the above integral exists.

Note 1 :-

L is a linear operation.

For $L[\alpha f(x) + \beta g(x)] = \int_0^\infty e^{-sx} [\alpha f(x) + \beta g(x)] dx$

$$= \alpha \int_0^\infty e^{-sx} f(x)\, dx + \beta \int_0^\infty e^{-sx} g(x)\, dx.$$

$$= \alpha L[f(x)] + \beta L[g(x)]$$

Note 2. $L[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right).$

Proof By definition

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)\, dx = F(s)$$

$$L[f(ax)] = \int_0^\infty e^{-sx} f(ax)\, dx$$

put

$$ax = y$$

$$\boxed{x = \frac{y}{a}}$$

$$\boxed{\frac{u}{v} = \frac{du\,v - u\,dv}{v^2}}$$ differentiation Rule.

$$x = \frac{y}{a}, \quad u = y, \quad v = a$$

$$dx = \frac{dy\,a - y(0)}{a^2}$$

$$= \frac{dy\,\cancel{a}}{a^{\cancel{2}}}$$

$$\boxed{dx = \frac{dy}{a}}$$

$$L[f(ax)] = \frac{1}{a}\int_0^\infty e^{-s\left(\frac{y}{a}\right)} f(y)\, dy$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)\,y} f(y)\, dy$$

$$\boxed{L[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)} \quad //.$$

Laplace tranform of Some Standard functions.

Result 1. $L(x^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

proof.

L.H.S. $L(x^n) = \int_0^{\infty} e^{-sx} x^n \, dx.$

$sx = y$

$\boxed{x = \frac{y}{s}.}$ $\quad u = y, \; v = s$

$dx = \frac{dy\, s - y(0)}{s^2}$

$dx = \frac{dy\, s}{s^2}$

$\boxed{dx = \frac{dy}{s}}$

$L(x^n) = \int_0^{\infty} e^{-y} \left(\frac{y}{s}\right)^n \frac{dy}{s}$

$= \int_0^{\infty} e^{-y} \frac{y^n}{s^n \cdot s} dy$

$\boxed{\Gamma n+1 = \int_0^{\infty} x^n e^{-x} dx.}$ $\quad = \int_0^{\infty} e^{-y} \cdot \frac{y^n}{s^{n+1}} dy$

$= \frac{1}{s^{n+1}} \int_0^{\infty} y^n e^{-y} dy$

$\boxed{L(x^n) = \frac{\Gamma(n+1)}{s^{n+1}}}$ By definition of Gamma function

Corollary

1) $L(x^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$

$\Gamma n+1 = n!$

$$\boxed{L(x^n) = \frac{n!}{s^{n+1}}}$$ when n is positive integer

2) $L(1) = \dfrac{1}{s}$

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)\, dx.$$

$$L(1) = \int_0^\infty e^{-sx}\, 1 \cdot dx.$$

$$= \lim_{m\to\infty} \int_0^m e^{-sx}\, dx.$$

$$= \left[\frac{e^{-sx}}{-s}\right]_0^m$$

$$= -\frac{e^{-s(\infty)}}{s} - \left[-\frac{e^{-s(0)}}{s}\right]$$

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$$= 0 + \frac{e^0}{s}$$

$$= 0 + \frac{1}{s}$$

$$\boxed{L(1) = \frac{1}{s}}$$

3) $L(x) = \dfrac{1}{s^2}$

$$L(x) = \int_0^\infty \underset{v}{e^{-sx}} \; \underset{u}{x} \, dx.$$

$$= \lim_{m \to \infty} \left[\left(-\frac{x}{s} e^{-sx} - \frac{1}{s^2} e^{-sx} \right) \right]_0^m$$

$$= \lim_{m \to \infty} \left[\left(-\frac{m}{s} e^{-sm} - \frac{1}{s^2} e^{-sm} \right) - \left[-\frac{0}{s} e^{-s(0)} - \frac{1}{s^2} e^{-s(0)} \right] \right.$$

$$= 0 - \left(0 - \frac{1}{s^2} \right)$$

$$\boxed{L(x) = 1/s^2.}$$

$$\int uv \, dx = u \int v \, dx - \int u' \left(\int v \, dx \right) dx.$$

3) $L(x) = \frac{1}{s^2}$.

L.H.S

$$L[f(x)] = \int_0^{\infty} e^{-sx} f(x)\, dx = F(s)$$

$$L(x) = \int_0^{\infty} x\, e^{-sx}\, dx.$$

Integration Rule

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$u = x \qquad dv = e^{-sx}\, dx.$

$du = dx \qquad v = \frac{e^{-sx}}{-s}$

$$= \lim_{m \to \infty} \left[x \frac{e^{-sx}}{-s} - \int \frac{e^{-sx}}{-s}\, dx. \right]_0^m$$

$$= \lim_{m \to \infty} \left[\frac{x}{-s} e^{-sx} + \frac{1}{s} \int e^{-sx}\, dx \right]_0^m$$

$$= \lim_{m \to \infty} \left[\frac{x}{-s} e^{-sx} + \frac{1}{s} \left\{ \frac{e^{-sx}}{-s} \right\} \right]_0^m$$

$$= \lim_{m\to\infty}\left[\frac{x e^{-sx}}{-s} - \frac{1}{s^2} e^{-sx}\right]_0^m$$

$$= \left[\left(\frac{m e^{-s(m)}}{-s} - \frac{1}{s^2} e^{-s(m)}\right) - \left[\frac{0 e^{-s(0)}}{-s} - \frac{1}{s^2} e^{-s(0)}\right]\right]$$

$$= \left[0 - 0 - \left(0 - \frac{1}{s^2} e^0\right)\right]$$

$$\boxed{\begin{aligned} e^0 &= 1 \\ e^{-\infty} &= 0. \end{aligned}}$$

$$\boxed{L(x) = \frac{1}{s^2}} \; //$$

4) $L(x^2) = \dfrac{2}{s^3}$

$$\cancel{L[f(x)] = \int_0^\infty e^{-sx} f(x)\,dx = F(s)}$$

$$L(x^2) = \int_0^\infty x^2 e^{-sx}\, dx.$$

$$L(x^n) = \frac{n!}{s^{n+1}}$$

$$L(x^2) = \frac{2!}{s^{2+1}} = \frac{2\times 1}{s^3} \; //$$

4) $L(x^2) = \frac{2}{s^3}.$

Solution

$$\boxed{L(x^n) = \frac{\Gamma n+1}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

$$L(x^2) = \frac{2!}{s^{2+1}}$$

$n=2$

$$= \frac{2\times1}{s^3}$$

$$\boxed{L(x^2) = \frac{2}{s^3}}$$ //.

5) $L(\sqrt{x}) = \frac{\sqrt{\pi}}{2s^{3/2}}.$

Solution.

$$\boxed{\begin{array}{l} L(x^n) = \frac{\Gamma n+1}{s^{n+1}} \\ \Gamma n+1 = n\Gamma n. \\ \Gamma \frac{1}{2} = \sqrt{\pi} \end{array}}$$

$L(x^{1/2})$

$n = 1/2.$

$$L(x^{1/2}) = \frac{\Gamma \frac{1}{2}+1}{s^{1/2+1}} \quad (\Gamma n+1)$$

$$= \frac{\frac{1}{2}\Gamma \frac{1}{2}}{s^{1/2+1}} \quad (n \Gamma n)$$

$$= \frac{\frac{1}{2} \times \sqrt{\pi}}{s^{\frac{1+2}{2}}}$$

$$\boxed{L(\sqrt{x}) = \frac{\sqrt{\pi}}{2 s^{3/2}}} \quad //.$$

Result 2 $\quad L(e^{ax}) = \frac{1}{s-a}$ if $s-a>0$.

Solution

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)\, dx = F(s)$$

$$L(e^{ax}) = \int_0^\infty e^{-sx} e^{ax}\, dx$$

$$= \int_0^\infty e^{-(s-a)x}\, dx.$$

$$= \lim_{m \to \infty} \int_0^m e^{-(s-a)x}\, dx.$$

$$= \lim_{m \to \infty} \left[-\frac{e^{-(s-a)x}}{(s-a)} \right]_0^m$$

$$= \lim_{m \to \infty} \left[-\frac{e^{-(s-a)m}}{s-a} - \left[-\frac{e^{-(s-a)0}}{s-a} \right] \right]$$

$$= \left[0 + \frac{e^{0}}{s-a} \right)$$

$$\boxed{L(e^{ax}) = \frac{1}{s-a} \quad \text{if } s-a>0}$$ //.

Corollary $L(e^{-ax}) = \frac{1}{s+a}$ if $s+a>0$.

Solution

$$L[f(x)] = \int_0^{\infty} e^{-sx} f(x)\,dx = F(s)$$

$$L(e^{-ax}) = \int_0^{\infty} e^{-sx} e^{-ax}\,dx$$

$$= \int_0^{\infty} e^{-(s+a)x}\,dx$$

$$= \lim_{m \to \infty} \int_0^{m} e^{-(s+a)x}\,dx.$$

$$= \lim_{m \to \infty} \left[\frac{e^{-(s+a)x}}{-(s+a)} \right]_0^{m}$$

$$= \lim_{m \to \infty} \left[- \frac{e^{-(s+a)x}}{(s+a)} \right]_0^m$$

$$= \lim_{m \to \infty} \left[- \frac{e^{-(s+a)m}}{(s+a)} - \left(- \frac{e^{-(s+a)0}}{s+a} \right) \right]$$

$$= \left[\frac{0}{s+a} + \frac{e^0}{s+a} \right]$$

$$\boxed{L(e^{-ax}) = \frac{1}{s+a} \quad \text{if } s+a > 0.}$$

//.

Result 3 $\quad L(\cos ax) = \dfrac{s}{s^2+a^2}.$

$$e^{ix} = \cos x + i \sin x$$

$$e^{iax} = \cos ax + i \sin ax$$

$$L(\cos ax) = \text{Real part of } \int_0^\infty e^{-ax} e^{iax} \, dx.$$

$$= \text{Real part of } \int_0^\infty L(e^{iax})$$

$$= \text{Real part of } \left(\frac{1}{s-ai} \right)$$

$$\boxed{L(e^{ax}) = \frac{1}{s-a} \quad \text{if } s-a > 0}$$

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$$= \text{Real part of } \left(\frac{1}{s-ai}\right) \times \frac{s+ai}{s}$$

$$= \text{Real part of } \left(\frac{s+ai}{s^2+a^2}\right)$$

$$= \frac{s}{s^2+a^2} + \frac{ai}{s^2+a^2}.$$

$$\boxed{L(\cos ax) = \frac{s}{s^2+a^2}.}$$

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Result 5 $\quad L(\cosh ax) = \dfrac{s}{s^2 - a^2}$

hyperbolic function.

$$\boxed{\cosh ax = \frac{e^{ax} + e^{-ax}}{2}.}$$

$$L(\cosh ax) = L\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

$$= \tfrac{1}{2} L(e^{ax}) + \tfrac{1}{2} L(e^{-ax})$$

$$\boxed{\begin{aligned} L(e^{ax}) &= \frac{1}{s-a} \\ L(e^{-ax}) &= \frac{1}{s+a}. \end{aligned}}$$

$$= \tfrac{1}{2}\left(\frac{1}{s-a}\right) + \frac{1}{2(s+a)}$$

$$= \frac{1}{2(s-a)} + \frac{1}{2(s+a)}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a)+(s-a)}{(s-a)\,(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right]$$

$$\boxed{L(\cosh ax) = \frac{s}{s^2-a^2}}$$

//

Result 6 $L(\sinh ax) = \left(\frac{a}{s^2-a^2}\right)$

hyperbolic function

$$\boxed{\begin{aligned} \sinh(x) &= \frac{e^{x}-e^{-x}}{2} \\ \sin(ax) &= \frac{e^{ax}-e^{-ax}}{2} \end{aligned}}$$

$$L(\sinh ax) = L\left(\frac{e^{ax}-e^{-ax}}{2}\right)$$

$$\boxed{L(e^{ax}) = \frac{1}{s-a} \quad \text{if } s-a>0}$$

$$\boxed{L(e^{-ax}) = \frac{1}{s+a} \quad \text{if } s+a>0}$$

$$= \frac{1}{2} L(e^{ax}) - \frac{1}{2} L(e^{-ax})$$

$$= \frac{1}{2}\left[\frac{1}{s-a}\right] - \frac{1}{2}\left[\frac{1}{s+a}\right]$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a) - (s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{s^2-a^2}\right]$$

$$\boxed{L(\sinh ax) = \frac{a}{s^2-a^2}}$$ //.

Result 7. $L[f'(x)] = sL[f(x)] - f(0)$

L.H.S $\quad L[f'(x)] = \int_0^\infty e^{-sx} f'(x)\, dx.$

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

$u = e^{-sx}$ $\qquad dv = f'(x)\,dx.$

$du = -s\,e^{-sx}$ $\qquad v = f(x)$

$$L[f'(x)] = \lim_{m\to\infty}\left\{\left[f(x)\,e^{-sx}\right]_0^m - \int_0^m f(x)(-s)\,e^{-sx}\,dx.\right.$$

$$L[f'(x)] = \lim_{m\to\infty}\left\{\left[f(m)\,e^{-sm} - f(0)\,e^{-s(0)}\right] + s\int_0^m e^{-sx} f(x)\,dx.\right\}$$

$$= 0 - f(0) + s\int_0^\infty e^{-sx} f(x)\,dx.$$

$$\boxed{L[f'(x)] = s\,L f(x) - f(0)}$$

<u>Result - 8</u> . $L[f''(x)] = s^2 L[f(x)] - s f(0) - f'(0)$

$L[f''(x)] = L[g'(x)]$ where $g(x) = f'(x)$

$$\boxed{L[f'(x)] = s\,L[f(x)] - f(0)}$$

$$L[f''(x)] = [s\,L[g(x)] - g(0)]$$

$$= s\,L[f'(x)] - f'(0)$$

$$= s[s\,L[f(x)] - f(0)] - f'(0)$$

$$\boxed{L[f''(x)] = s^2 L[f(x)] - s f(0) - f'(0)}$$

<u>Note</u> In general

$$L[f^n(x)] = s^n L[f(x)] - s^{n-1} f(0) - s^{n-2} f'(0) - \dots f^{n-1}(0)$$

Prove.

Result 9 If $L[f(x)] = F(s)$ then $L[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

proof
$$L[f(ax)] = \int_0^\infty e^{-sx} f(ax)\, dx.$$

Put $ax = y$

$$x = \frac{y}{a} \quad \Rightarrow u \quad \Rightarrow v$$

$$\frac{u}{v} = \frac{d\,uv - u\,dv}{v^2}.$$

$$dx = \frac{dy\,a - y(0)}{a^2} = \frac{dy\,a}{a^2}$$

$$\boxed{dx = \frac{dy}{a}.}$$

$$L[f(ax)] = \int_0^\infty e^{-s\left(\frac{y}{a}\right)} f(y) \frac{dy}{a}.$$

$$= \int_0^\infty e^{-(s/a)y} f(y) \frac{dy}{a}.$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)y} f(y)\, dy$$

$$\boxed{L[f(ax)] = \left(\frac{1}{a}\right) F\left(\frac{s}{a}\right)}$$

Result 10 If $L[f(x)] = F(s)$ then

(i) $L[e^{-ax} f(x)] = F(s+a)$

(ii) $L[e^{ax} f(x)] = F(s-a)$

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Solut

(i) $L[e^{-ax} f(x)] = \int_0^\infty e^{-sx} e^{-ax} f(x)\, dx.$

$= \int_0^\infty e^{-(s+a)x} f(x)\, dx.$

$$\boxed{L[e^{-ax} f(x)] = F(s+a)}$$

(ii) $L[e^{ax} f(x)] = \int_0^\infty e^{-sx} e^{ax} f(x)\, dx.$

$= \int_0^\infty e^{-(s-a)x} f(x)\, dx.$

$$\boxed{L[f(x)] = \int_0^\infty e^{-sx} f(x)\, dx} = F(s)$$

$$\boxed{L[e^{ax} f(x)] = F(s-a)}$$

Examples

1) we know that

$L(1) = \frac{1}{s}$ Hence $L\, e^{(-ax)} = \frac{1}{s+a}.$

$L(\cos bx) = \frac{s}{s^2+b^2}$

Hence $L(e^{-ax} \cos bx) = \frac{s+a}{(s+a)^2+b^2}.$

$\because \boxed{s = s+a}.$

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2) we know that $L(x^n) = \frac{n!}{s^{n+1}}$

Hence $L(e^{ax} x^n) = \frac{n!}{(s-a)^{n+1}}$

$$\boxed{L(e^{ax}) = \frac{1}{s-a}}$$

Result 11. If $L[f(x)] = F(s)$ then

$$L[x f(x)] = -\frac{d}{ds}[F(s)]$$

Proof.

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-sx} f(x)\,dx.$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-sx} f(x)\right] dx.$$

$$= \int_0^\infty -x e^{-sx} f(x)\,dx.$$

$$= -\int_0^\infty e^{-sx} x f(x)\,dx$$

$$\frac{d[F(s)]}{ds} = -L[x f(x)]$$

$$\boxed{\therefore L[x f(x)] = -\frac{d}{ds}[F(s)]}$$ //.

Note: In general

$$\boxed{L\,(x^n f(x)] = (-1)^n \frac{d^n}{ds^n}\left(L[f(x)]\right)}$$

Solved Problems:

Problem 1:- Find the Laplace Transform of

$t^2 + \cos 2t \cos t + \sin^2 2t$

Solution

$$\boxed{\begin{aligned} \sin^2(x) &= \frac{1}{2}\left[1-\cos 2x\right] \\ \cos(x)\cos(y) &= \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right] \end{aligned}}$$

$$\boxed{\sin^2 2t = \frac{1}{2}\left[1-\cos 4x\right]}$$

$$\cos 2t \cos t = \frac{1}{2}\left[\cos(2t+t)+\cos(2t-t)\right]$$

$$\boxed{\cos 2t \cos t = \tfrac{1}{2}\left[\cos 3t + \cos t\right]}$$

$$\begin{aligned} L\left(t^2+\cos 2t\cos t + \sin^2 2t\right) &= L(t^2) + L(\cos 2t \cos t) \\ &\quad + L(\sin^2 2t) \\ &= L(t^2) + L\left[\tfrac{1}{2}(\cos 3t + \cos t)\right] + \\ &\quad L\left(\tfrac{1}{2}(1-\cos 4t)\right] \\ &= L(t^2) + \tfrac{1}{2}\left[L(\cos 3t) + L(\cos t)\right] + \\ &\quad \tfrac{1}{2}\left[L(1) - L(\cos 4t)\right] \end{aligned}$$

$L(x^n) = \frac{n!}{s^{n+1}}$

$L(t^2) = \frac{2\times 1}{s^{2+1}}$

$=$

$$\boxed{L(t^2) = \frac{2}{s^3}}$$

$$\boxed{L(\cos ax) = \frac{s}{s^2+a^2}}$$

$L(\cos 3t) = \frac{s}{s^2+3^2} = \frac{s}{s^2+9}.$

$L(\cos t) = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}$

$$\boxed{L(1) = \frac{1}{s}}$$

$L(\cos 4t) = \frac{s}{s^2+4^2} = \frac{s}{s^2+16}.$

$= \frac{2}{s^3} + \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right] + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$ //

INVERSE LAPLACE TRANSFORM

Some important results in Inverse LT :-

Result 1 :- $L^{-1}[F(s+a)] = e^{-ax} L^{-1}[F(s)]$

Proof Let $L[f(x)] = F(s)$

Then we know

$$L[e^{-ax} f(x)] = F(s+a)$$

Hence $L^{-1}[F(s+a)] = e^{-ax} f(x)$

$$\boxed{L^{-1} F(s) = f(x)}$$

$$\boxed{L^{-1}[F(s+a)] = e^{-ax} L^{-1}[F(s)]}$$

Result 2 $L^{-1}[F(\lambda s)] = \frac{1}{\lambda} F\left(\frac{x}{\lambda}\right)$

Proof we know $L[f(ax)] = (1/a) F(s/a)$

By Result 9

$L[f(ax)] = 1/a$ 1

$\therefore L^{-1} (1/a) F(s/a) = f(ax)$

put $1/a = \lambda$. Then we have

$$L^{-1}[\lambda F(\lambda s)] = F(x/\lambda)$$

Hence $\boxed{L^{-1}[F(\lambda s)] = \frac{1}{\lambda} F\left(\frac{x}{\lambda}\right)}$

Result 3 $L^{-1}[F'(s)] = -x\, L^{-1}[F(s)]$

Proof: we know that $L[x f(x)] = (-)^n \frac{d}{ds}[F(s)]$

$= -F'(s)$

$\therefore L^{-1}[F'(s)] = -x f(x)$

$$\boxed{L^{-1}[F(s)] = f(x)}$$

$$\boxed{L^{-1}[F'(s)] = -x\, L^{-1}[F(s)]}$$

Solved problems

Problem 1. Find the inverse Laplace Transform for

(i) $\frac{1}{(s+3)^2+25}$ (ii) $\frac{s}{(s+2)^2}$

(iii) $\frac{s+1}{s^2+2s+2}$ (iv) $\frac{s}{a^2s^2+b^2}$

Solution. (i) $\frac{1}{(s+3)^2+25}$

$$L^{-1}\left[\frac{1}{(s+3)^2+25}\right] = e^{-3x} L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$= e^{-3x} L^{-1}\left[\frac{5}{5(s^2+5^2)}\right]$$

$$= \frac{1}{5} e^{-3x} L^{-1}\left[\frac{5}{s^2+5^2}\right]$$

$$\boxed{L[\sin ax] = \frac{1}{s^2+a^2}}$$

$e^{-ax} \sin$

(i) $\frac{1}{(s+3)^2+25}$

Solution

$$\boxed{L[e^{-ax} \sin bx] = \frac{b}{(s+a)^2+b^2}.}$$

$$L^{-1}\left[\frac{1}{(s+3)^2+25}\right] = e^{-3x} L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$= e^{-3x} L^{-1}\left[\frac{5}{5(s^2+5^2)}\right]$$

$$= \frac{1}{5} e^{-3x} L^{-1}\left[\frac{5}{s^2+5^2}\right]$$

$$\boxed{L^{-1}\left[\frac{1}{(s+3)^2+25}\right] = \frac{1}{5} e^{-3x} \sin 5x}$$

(ii) $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Solution

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{s+\cancel{x}}{s^2+\cancel{4}2}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s+2}\right] - 2\,L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$\boxed{e^{-ax} = \frac{1}{s+a}} \qquad \boxed{x e^{-ax} = \frac{1}{(s+a)^2}}$$

$$= e^{-2x} - 2\,e^{-2x}\,x$$

$$\boxed{L^{-1}\left[\frac{s}{(s+2)^2}\right] = e^{-2x}\left[1 - 2x\right]}$$ //.

(iii) $L^{-1}\left(\frac{s+1}{s^2+2s+2}\right)$

$$L^{-1}\left[\frac{s+1}{s^2+2s+2}\right] = L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$$

$$\boxed{e^{-ax}\cos bx = \frac{s+a}{(s+a)^2+b^2}.}$$

$$\boxed{L^{-1}\left[\frac{s+1}{s^2+2s+2}\right] = e^{-x}\cos x}$$ ///.

Unit – IV

Numerical Differentiation – Derivatives using Newton's Forward Difference formula – Derivatives using Newton's Backward Difference formula – Derivatives using Newton's Central difference formula – Maxima and Minima of the interpolating polynomial.

UNIT IV

NUMERICAL DIFFERENTIATION.

Numerical differentiation :- → Finding derivatives

*. The process of computing the derivative $\frac{dy}{dx}$ for some particular value of x is called numerical differentiation.

Numerical integration :-

*. The process of evaluating the definite integral $\int_a^b f(x)\,dx$ is called "Numerical integration".

Application of Derivatives.

*. We use the derivative to determine the maximum and minimum values of particular functions eg cost, strength, amount of material used in building. profit, loss, etc.

*. Derivatives are met in many engineering and science problems, especially when modelling the behaviour of moving objects.

what are the basic differentiation rules:-

*. The sum rule says the derivative of a Sum of functions is the sum of their derivatives.

*. The difference rule says the derivative of a difference functions is the difference of their derivatives.

1) Derivatives using Newton's forward difference formula.

2. Derivatives using Newton's backward difference formula

3) Derivatives using central difference Formula.

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Derivatives using Newton's forward difference formula :-

Newton's interpolation formula for equal interval is.

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots \rightarrow (1)$$

where $p = \frac{x - x_0}{h}$ $\rightarrow (2)$.

Differentiating (1) w.r.t. p we get

↳ with respect to

$\cancel{\frac{dy}{dp}}$ ≠ Differentiating using power Rule.

$$\boxed{\left(\frac{d}{dx}\right)(x^n) = n\,x^{n-1}}$$

eg $\left(\frac{d}{dx}\right)(p^2) = 2p^{2-1}$
$= 2p$ //.

$$\frac{dy}{dp} = \Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{(p^2-p)(p-2)}{3!}\Delta^3 y_0 + \frac{(p^2-p)(p^2-3p-2p+6)}{4!}\Delta^4 y_0 + \dots$$

$$\frac{dy}{dp} = \Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \frac{p^3 - 2p^2 - p^2 + 2p}{3!}\Delta^3 y_0 + \frac{p^4 - 5p^3 + 6p^2 - p^3 + \cancel{6p} - \cancel{6p}}{4!}\Delta^4 y_0 + \dots$$

$3p^2$.

$$\frac{dy}{dp} = \Delta y_0 + \left(\frac{2p-1}{2!}\right) \Delta^2 y_0 + \left(\frac{p^3 - 3p^2 + 2p}{3!}\right) \Delta^3 y_0 +$$

$$\left(\frac{p^4 - 6p^3 + 6p^2}{4!}\right) \Delta^4 y_0 + \cdots\cdots$$

$$\frac{dy}{dp} = \Delta y_0 + \frac{(2p-1)}{2!} \Delta^2 y_0 + \left(\frac{3p^2 - 6p + 2}{3!}\right) \Delta^3 y_0 +$$

$$\frac{4p^3 - 18p^2 + 12p}{4}$$

First simply the following part.

$$\left(\frac{p(p-1)}{2!}\right) \Rightarrow \left(\frac{p^2 - p}{2!}\right)$$

$$\left(\frac{p(p-1)(p-2)}{3!}\right) \Rightarrow \frac{(p^2 - p)(p-2)}{3!}$$

$$\Rightarrow \frac{p^3 - 2p^2 - p^2 + 2p}{3!}$$

$$= \left(\frac{p^3 - 3p^2 + 2p}{3!}\right) \;//$$

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$$\left(\frac{p(p-1)(p-2)(p-3)}{4!}\right) \Rightarrow \frac{(p^2-p)(p-2)(p-3)}{4!}$$

$$\Rightarrow \frac{(p^3-2p^2-p^2+2p)(p-3)}{4!}$$

$$\Rightarrow \frac{(p^3-3p^2+2p)(p-3)}{4!}$$

$$\Rightarrow \frac{p^4-3p^3-3p^3+9p^2+2p^2-6p}{4!}$$

$$\Rightarrow \left(\frac{p^4-6p^3+11p^2-6p}{4!}\right) //.$$

$$y(x) = y_0 + p\Delta y_0 + \left(\frac{p^2-p}{2!}\right)\Delta^2 y_0 + \left(\frac{p^3-3p^2+2p}{3!}\right)\Delta^3 y_0$$

$$+\left(\frac{p^4-6p^3+11p^2-6p}{4!}\right)\Delta^4 y_0.$$

Differentiating (1) with respect to p we get.

$$\frac{dy}{dp} = \Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{3!}\right)\Delta^3 y_0 +$$

$$\left(\frac{4p^3-18p^2+22p-6}{4!}\right)\Delta^4 y_0.$$

$$\Bigg[\ \frac{(p^2-p)}{2!} = \frac{2p^{2-1}-p^{1-1}}{2!}$$

$$= \left(\frac{2p-p^0}{2!}\right) \Rightarrow \left(\frac{2p-1}{2!}\right)$$

(Any power value 0 is equal to 1)

$$\left(\frac{p^3-3p^2+2p}{3!}\right) = \frac{3p^{3-1}-3\times 2p^{2-1}+2p^{1-1}}{3!}$$

$$=\left(\frac{3p^2-6p+2p^0}{3!}\right)$$

$$=\left(\frac{3p^2-6p+2}{3!}\right)$$

$$\left(\frac{p^4-6p^3+11p^2-6p}{4!}\right) =$$

$$= \frac{4p^{4-1}-6\times 3p^{3-1}+11\times 2p^{2-1}-6p^{1-1}}{4!}$$

$$=\frac{4p^3-18p^2+22p+6p^0}{4!}$$

$$=\left(\frac{4p^3-18p^2+22p+6}{4!}\right)$$

Differentiating (2) w.r.t x we have

$$p = \frac{x - x_0}{h}$$

$$\boxed{\frac{dp}{dx} = \frac{1}{h}}$$

Now $\boxed{\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}}$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=x_0+ph} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{3!}\right)\Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{4!}\right)\Delta^4 y_0 + \cdots\right]}$$

$x = x_0 + ph \Rightarrow$ [$p = \frac{x - x_0}{h}$

$ph = x - x_0$

$x = x_0 + ph$)

At $x = x_0$, $p = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{2(0)-1}{2!}\Delta^2 y_0 + \frac{3(0)^2-6(0)+2}{3!} + \frac{4(0)^3-18(0)^2+22(0)-6}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{\cancel{2}}{\underset{1\times\cancel{2}\times 3}{3!}}\Delta^3 y_0 + \frac{\cancel{6}}{\underset{1\times\cancel{2}\times\cancel{3}\times 4}{4!}}\Delta^4 y_0 + \dots\right]$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 + \dots\right]}$$

Now $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{d}{dp}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{1}{h}.}$

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left[\frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{3!}\right)\Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{4!}\right)\Delta^4 y_0.\right]\right]$$

$$= \frac{1}{h^2}\left[\frac{\cancel{2}}{\cancel{2}\times 1}\Delta^2 y_0 + \left(\frac{6p-6}{3\times 2\times 1}\right)\Delta^3 y_0 + \left(\frac{12p^2-36p+11}{4\times 3\times 2\times 1}\right)\Delta^4 y_0 \dots\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{\cancel{6}(p-1)}{\cancel{6}}\Delta^3 y_0 + \frac{\cancel{2}(6p^2-18p)+11}{4\times3\times\cancel{2}\times1}\Delta^4 y_0 + \cdots\right]$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right]}$$

At $x = x_0$, $p = 0$.

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{6(0)-18(0)+11}{12}\Delta^4 y_0 + \cdots\right]$$

$\frac{1}{h}$.

$$\boxed{\therefore \left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]}$$

Derivatives of higher order can similarly be obtained.

Problem - 1

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 51$ from the following data

	x_0	x_1	x_2	x_3	x_4
x	50	60	70	80	90
y	y_0 19.96	y_1 36.65	y_2 58.81	y_3 77.21	y_4 94.61

Solution :-

Here $h = 10$. To find the derivatives of y at $x = 51$, we use Newton's forward formula taking the origin at $x = 50$.

we have $p = \frac{x - x_0}{h} = \frac{51 - 50}{10}$

$$\boxed{p = 0.1}$$

$\therefore$ At $x = 51$, $p = 0.1$,

The difference table is given by.

x	$P = \frac{x-50}{10}$	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
50	$P = \frac{50-50}{10}$ $\boxed{P = 0}$	19.96				
			16.69			
60	$P = \frac{60-50}{10} = \frac{\cancel{10}}{\cancel{10}}$ $\boxed{P=1}$	36.65		5.47		
			22.16		-9.23	
70	$P = \frac{70-50}{10} = \frac{20}{10}$ $\boxed{P=2}$	58.81		-3.76		11.99.
			18.4		2.76	
80	$P = \frac{80-50}{10} = \frac{3\cancel{0}}{1\cancel{0}}$ $\boxed{P=3}$	77.21		-1.00		
			17.4			
90	$P = \frac{90-50}{10} = \frac{4\cancel{0}}{3\cancel{0}}$ $\boxed{P=4}$	94.61				

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$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{P=0.1} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2P-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3P^2-6P+2}{3!}\right)\Delta^3 y_0 + \frac{4P^3-18P^2+22P-6}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\therefore \frac{1}{10}\left[16.69 + \left(\frac{2(0.1)-1}{2!}\right)(5.47) + \left(\frac{3(0.1)^2-6(0.1)+2}{3\times2\times1}\right) \times(-9.23) + \frac{4(0.1)^3-18(0.1)^2+22(0.1)-6}{2}\right]$$

$$\left(\frac{dy}{dx}\right)_{P=0.1} = \frac{1}{10}\left[16.69 + \left(\frac{2(0.1)-1}{2\times1}\right)(5.47) + \left(\frac{3(0.1)^2-6(0.1)+2}{3\times2\times1}\right)\times(-9.23) + \left(\frac{4(0.1)^3-18(0.1)^2+22(0.1)-6}{4\times3\times2\times1}\right)\times 11.99\right]$$

$$= \frac{1}{10}\left[16.69 + \left(\frac{0.2-1}{2}\right)(5.47) + \left(\frac{0.03-0.6+2}{6}\right)\times -9.23 + \frac{0.004-0.18+2.2-6}{24}\times 11.99 + \cdots\right]$$

$$= \frac{1}{10}\left[16.69 - 0.4\times 5.47 + 0.23833\times -9.23 + (-0.165666)\times 11.99 + \cdots\right]$$

$$= \frac{1}{10}\left[16.69 - 2.188 - 2.1998 - 1.9863 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{p=0.1} = \frac{1}{10} \times 10.3159 = 1.03159$$

$$=$$

$$\boxed{\left(\frac{dy}{dx}\right)_{p=0.1} = 1.0316}$$

$$\left(\frac{d^2y}{dx^2}\right)_{p=0.1} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \dots\right]$$

$$= \frac{1}{(10)^2}\left[5.47 + (0.1-1)(-9.23) + \left(\frac{6(0.1)^2-(18\times 0.1)+11}{12}\right)\times 11.99 + \dots\right]$$

$$= \frac{1}{100}\left[5.47 - 0.9 \times -9.23 + \frac{0.06 - 1.8 + 11}{12} \times 11.99 + \dots\right]$$

$$= \frac{1}{100}\left[5.47 + 8.307 + 0.77166 \times 11.99\right]$$

$$= \frac{1}{100}\left[5.47 + 8.307 + 9.2523\right]$$

$$= \frac{1}{100}\left[23.0293\right]$$

$$\boxed{\left[\frac{d^2y}{dx^2}\right]_{p=0.1} = 0.2303}$$

19-08-2020

Solved problem: 2.

Find $y'(x)$ given

x	0	1	2	3	4
$y(x)$	1	1	15	40	85

Hence find $y'(x)$ at $x=0.5$.

Solution:-

Here $h=1$. We apply Newton's forward difference for derivative.

$$y'_p = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{3!}\right)\Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{4!}\right)\Delta^4 y_0 + \dots\right]$$

where $p = \frac{x-x_0}{h}$, choose $x_0 = 0$.

$$p = \frac{0.5-0}{1}$$

$\boxed{p = 0.5}$ so that $p = x$.

The difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		0			
1	1		14	-3	
		14			12
2	15		11		$(9-(-3))$
		25		9	
3	40		20		
		45			
4.	85				

$$y' = \Delta y_0 + \left(\frac{2x-1}{2\times 1}\right)\Delta^2 y_0 + \left(\frac{3x^2-6x+2}{3\times2\times1}\right)\Delta^3 y_0 + \left(\frac{4x^3-18x^2+22x-6}{4\times3\times2\times1}\right)\Delta^4 y_0 + \cdots$$

$$= 0 + \frac{2x-1}{\cancel{2}} \times \overset{7}{\cancel{14}} + \left(\frac{3x^2-6x+2}{\underset{2}{\cancel{6}}}\right) \times -\cancel{3} + \left(\frac{4x^3-18x^2+22x-6}{\underset{2}{\cancel{24}}} \times \cancel{12}\right)$$

$$= 7(2x-1) - \left(\frac{3x^2-6x+2}{2}\right) + \left(\frac{4x^3-18x^2+22x-6}{2}\right)$$

$$= 14x - 7 + \frac{4x^3-21x^2+28x-8}{2}$$

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$$= \frac{28x - 14 + 4x^3 - 21x^2 + 28x - 8}{2}.$$

$$= \frac{4x^3 - 21x^2 + 56x - 22}{2}$$

$$= \frac{\cancel{2}(2x^3 + 28x - 11)}{\cancel{2}} - \frac{21x^2}{2}$$

$$\boxed{\therefore y'(x) = 2x^3 - \frac{21x^2}{2} + 28x - 11}$$

Now y' at $x = 0.5$ is $y'(0.5)$

$$y'(0.5) = 2(.5)^3 - 10.5(.5)^2 + 28(.5) - 11$$

$$= 2 \times 0.125 - 10.5 \times 0.25 + 14 - 11$$

$$= 0.25 - 2.625 + 14 - 11$$

$$\boxed{y'(0.5) = 0.625}$$ //.

21/8/2020

Derivatives using Newton's Backward difference formula.

⇒ We know that Newton's interpolation formula for backward differences is.

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \cdots \rightarrow (1)$$

where $p = \frac{x - x_n}{h}$

As before, differentiating (1) :

$$\frac{p(p+1)}{2!} = \frac{p^2+p}{2!}$$

$$\frac{p(p+1)(p+2)}{3!} \Rightarrow \frac{(p^2+p)(p+2)}{3!}$$

$$\Rightarrow \frac{p^3+2p^2+p^2+2p}{3!}$$

$$\Rightarrow \left(\frac{p^3+3p^2+2p}{3!}\right)$$

$$\frac{p(p+1)(p+2)(p+3)}{4!} \Rightarrow \frac{(p^3+3p^2+2p)(p+3)}{4!}$$

$$\Rightarrow \frac{p^4+3p^3+3p^3+9p^2+2p^2+6p}{4!}$$

$$= \left(\frac{p^4+6p^3+11p^2+6p}{4!}\right)$$

$$y_p = y_n + p\nabla y_n + \frac{p^2+p}{2\times 1}\nabla^2 y_n + \left(\frac{p^3+3p^2+2p}{3\times 2\times 1}\right)\nabla^3 y_n + \frac{p^4+6p^3+11p^2+6p}{4\times 3\times 2\times 1}\nabla^4 y_n + \cdots \longrightarrow (2)$$

Differentiating (2) w.r.t p we have.

$$\frac{dy}{dp} = \nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \left(\frac{4p^3+18p^2+22p+6}{24}\right)\nabla^4 y_n + \cdots$$

$$\frac{dy}{dp} = \nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \frac{\cancel{2}(2p^3+9p^2+11p+3)}{\cancel{24}\ 12}\nabla^4 y_n + \cdots$$

$$\frac{dy}{dp} = \nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_n + \cdots \longrightarrow (3)$$

Differentiating $p = \frac{x - x_n}{h}$ w.r.t x we get.

$$\boxed{\frac{dp}{dx} = \frac{1}{h}}$$

Now $$\boxed{\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}}$$

$$\boxed{\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_n + \cdots\right]}$$

At $x = x_n$, $p = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2(0)+1}{2}\right)\nabla^2 y_n + \left(\frac{3(0)^2+6(0)+\cancel{2}}{\cancel{6}\,3}\right)\nabla^3 y_n + \left(\frac{2(0)^3+9(0)^2+11(0)+\cancel{3}}{\cancel{12}\,4}\right)\nabla^4 y_n + \cdots\right]$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]}$$

Now $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{d}{dp}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{1}{h}}$

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left[\frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \left(\frac{2p^3+9p^2+11p+3}{12}\right)\nabla^4 y_n + \cdots\right]\right]\cdot\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\frac{\cancel{2}}{\cancel{2}}\nabla^2 y_n + \left(\frac{6p+6}{6}\right)\nabla^3 y_n + \left(\frac{6p^2+18p+11}{12}\right)\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \left(\frac{\cancel{6}(p+1)}{\cancel{6}}\right)\nabla^3 y_n + \left(\frac{6p^2+18p+11}{12}\right)\nabla^4 y_n + \cdots\right]$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \left(\frac{6p^2+18p+11}{12}\right)\nabla^4 y_n + \cdots\right]}$$

At $x = x_n$, $p = 0$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + (0+1)\nabla^3 y_n + \left(\frac{6(0)^2 + 18(0) + 11}{12}\right) + \nabla^4 y_n + \cdots\right]$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]}$$

Similarly we can find the higher order derivatives.

Solved Problem: 4

The population of a certain town is shown in the following table.

Year x	1931	1941	1951	1961	1971
Population y	40.62	60.80	79.95	103.56	132.65

Finding the rate of growth of the population in 1961.

Solution.

Here $h = 10$. Since the rate of growth of population is $\frac{dy}{dx}$ we have to find $\frac{dy}{dx}$ at $x = 1961$, which lies nearer to the end value of the table.

*. Hence we choose the origin at $x = 1971$

and we ~~have~~ use Newton's backward difference formula for derivative.

$n = 4$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_4 + \left(\frac{2p+1}{2}\right)\nabla^2 y_4 + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_4 + \left(\frac{2p^3+9p^2+11p+3}{12}\right)\nabla^4 y_4 + \cdots\right]}$$

$$p = \frac{x - x_n}{h} = \frac{1961-1971}{10}$$

$$p = -\frac{\cancel{10}}{\cancel{10}}$$

$$\boxed{p = -1}$$

The backward difference table.

YEAR x	Population y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1931	40.62				
		20.18			
			-1.03		
1941	60.80			5.49	
		19.15			
			4.46		-4.47
1951	79.95				
		23.61		1.02	
			5.48		
1961	103.56				
		29.09			
1971	132.65				

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$$\left(\frac{dy}{dx}\right)_{p=-1} = \frac{1}{10}\left[29.09 + \left(\frac{2(-1)+1}{2}\right) * 5.48 + \frac{3(-1)^2+6(-1)+2}{6} \times 1.02 + \frac{2(-1)^3+9(-1)^2+11(-1)+3}{12} * (-4.47)\right)$$

$$= \frac{1}{10}\left[29.09 - \frac{\cancel{2}+1}{\cancel{2}} * 5.48 + \left(\frac{3-6+2}{6}\right)(1.02) + \frac{-2+9-11+3}{12} * (-4.47)\right)$$

$$= \frac{1}{10}\left[29.09 - \frac{1}{2} * 5.48 - \frac{1}{6} * 1.02 - \frac{1}{12} * (-4.47)\right]$$

$$= \frac{1}{10}\left[29.09 - 2.74 - 0.17 + 0.3725\right]$$

$$= \frac{1}{10}\left[26.5525\right]$$

$$\boxed{\left(\frac{dy}{dx}\right)_{p=-1} = 2.6553}$$

Derivatives using central difference formula.

Derivatives using Stirling's formula :-

The Stirling's formula is

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_1}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}$$

$$\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-2^2)}{5!}$$

$$\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \cdots \longrightarrow (1)$$

where $p = \dfrac{x - x_0}{h} \longrightarrow (2)$

we want.

$$\boxed{\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}.} \longrightarrow (3)$$

Diffentiating equation No (1) we get

dy and differentiating equation No (2) we get

$\frac{dp}{dx}$.

Before differentiating the equation No (1) we need to simplify the followings terms.

$$\frac{p(p^2-1^2)}{3!} = \frac{p(p^2-1)}{3!} \Rightarrow \left(\frac{p^3-p}{3!}\right).$$

$$\frac{p^2(p^2-1)}{4!} \Rightarrow \left(\frac{p^4-p^2}{4!}\right)$$

$$\left(\frac{p(p^2-1)(p^2-2^2)}{5!}\right) = \left(\frac{(p^3-p)(p^2-4)}{5!}\right)$$

$$= \left(\frac{p^5-4p^3-p^3+4p}{5!}\right)$$

$$= \left(\frac{p^5-5p^3+4p}{5!}\right)$$

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}(\Delta^2 y_{-1}) + \left(\frac{p^3-p}{3!}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{p^4-p^2}{4!}\right)\Delta^4 y_{-2} + \left(\frac{p^5-\cancel{4}5p^3+4p}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \dots \rightarrow (1).$$

Differentiating (1) w.r.t p we get. using power rule.

$$\left(\frac{dy}{dp}\right) = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\,\Delta^2 y_{-1} + \left(\frac{3p^2-1}{3!}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \left(\frac{5p^4 - 15p^2 + 4}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \cdots \rightarrow (4)$$

Differentiating (2) w.r.t x we get

$$p = \frac{x - x_0}{h}$$

$$\boxed{\frac{dp}{dx} = \frac{1}{h}} \rightarrow (5)$$

Substitution the equation (4) & (5) in equation No (3) we get the following equation.

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\,\Delta^2 y_{-1} + \frac{(3p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{5p^4 - 15p^2 + 4}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \cdots\right]}$$

At $x = x_0,\ p = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + (0)\,\Delta^2 y_{-1} + \left(\frac{3(0)^2-1}{3!}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{5(0)^4 - 15(0)^2 + 4}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right.$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{\cancel{4}}{5\times\cancel{4}\times 3\times 2\times 1} \times \left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right]$$

Similary we can derive the Second order derivation is

$$\boxed{\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right) \times \frac{1}{h}}$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h} \times \frac{1}{h}\left[\Delta^2 y_{-1} + \frac{6p}{3\times 2\times 1}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{12p^2 - 2}{4!}\right)\Delta^4 y_{-2} + \left(\frac{20p^3 - 30p}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_{-1} + \frac{\cancel{6}p}{\cancel{6}}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \right.$$

$$\left(\frac{\cancel{2}(6p-1)}{4\times 3\times \cancel{2}\times 1}\right)\Delta^4 y_{-1} + \frac{\cancel{10}\,2\,(2p^3 - 3p)}{\cancel{5}\times 4\times 3\times \cancel{2}\times 1} \times \left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + p\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \right.$$

$$\left.\left(\frac{6p-1}{12}\right)\Delta^4 y_{-1} + \left(\frac{2p^3-3p}{12}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right]$$

At $x = x_0$, $p = 0$.

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + (0)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \right.$$

$$\left.\left(\frac{6(0)-1}{12}\right)\Delta^4 y_{-1} + \left(\frac{2(0)^3 - 3(0)}{12}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right]$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\left(\Delta^4 y_{-1}\right) + \ldots\right]}$$

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Unit IV.

8.4. Maxima and minima of the interpolating polynomial

Problem 8. Find the maximum and minimum value of y from the following table.

x	0	1	2	3	4	5
y	0	$1/4$	0	$9/4$	16	$\frac{225}{4}$

Solution. Here $h=1$.

Newton's forward difference formula is.

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0 + \frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y_0 + \cdots\right]}$$

Forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0					
		0.25				
1	0.25		-0.50			
		-0.25		3		
2	0		2.50		6	
		2.25		9		0
3	2.25		11.50		6	
		13.75		15		
4	16		26.50			
		40.25				
5	56.25					

Choosing the origin at $x_0=0$, $p = \frac{x-0}{1} = x$.

$$\frac{dy}{dx} = \frac{1}{1}\left[0.25 + \frac{2p-1}{2} \times (-0.50) + \frac{3p^2-6p+2}{3\times 2\times 1} \times \cancel{3} + \frac{4p^3-18p^2+22p-6}{4\times \cancel{3}\times \cancel{2}\times 1} \times \cancel{6} + \cdots\right]$$

$$\frac{dy}{dx} = 0.25 - \frac{p+0.50}{2} + \frac{3p^2-6p+2}{2} + \frac{4p^3-18p^2+22p-6}{4}$$

$$\frac{dy}{dx} = \frac{1}{4} - \frac{2p+1}{4} + \frac{6p^2-12p+4}{4} + \frac{4p^3-18p^2+22p-6}{4}$$

$$= \frac{4p^3-18p^2+6p^2+22p-2p-12p+1+1+4-6}{4}$$

$$\frac{dy}{dx} = 4p^3-12p^2+8p.$$

Now $\frac{dy}{dx} = 0 \Rightarrow 4p^3-12p^2+8p = 0$

$$\Rightarrow 4p^3-12p^2+8p = 0$$

$$\Rightarrow 4p(p-2)(p-1) = 0$$

$$4p = 0$$

$$p-2 = 0$$

$$p-1 = 0.$$

$$\Rightarrow p = 0, 1, 2.$$

Also $\frac{d^2y}{dx^2} = \frac{d}{}\left(\frac{dy}{dx}\right)$

$$= d\,(4p^3 - 12p^2 + 8p)$$

$$\frac{d^2y}{dx^2} = 12p^2 - 24p + 8$$

At $p = 0$ $\quad = 12(0) - 24(0) + 8$

$\boxed{\frac{d^2y}{dx^2} = 8}$ which is positive

At $P=1$, $\frac{d^2y}{dx^2} = 12(1) - 24(1) + 8$

$= 12 - 24 + 8$

$\boxed{\frac{d^2y}{dx^2} = -4}$ which is negative.

At $P=2$, $\frac{d^2y}{dx^2} = 12(2)^2 - 24(2) + 8$

$= 12(4) - 48 + 8$

$= 48 - 48 + 8$

$\boxed{\frac{d^2y}{dx^2} = 8}$ which is positive.

$\therefore$ y is maximum at $P=1$ & minimum at $P=0$ & 2.

$\therefore$ The maximum value y at $P=1$ $(x=1)$ is

$$y(x) = \frac{1}{h}\left[y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \dots\right]$$

$$y(1) = \frac{1}{1}\left[0 + 1(0.25) + \frac{1(1-1)}{2} \times -0.50 + \frac{1(1-1)(1-2)}{6!} \times 3 + \dots\right]$$

$y(1) = 0 + 1(0.25)$

$\boxed{y(1) = 0.25}$ Maximum.

Minimum at $x = 0$ & $x = 2$.

$$y(0) = \frac{1}{1}\left[0 + 0(0.25) + \frac{0(0-1)}{2} \times -0.50 + \cdots\right]$$

$\boxed{y(0) = 0}$ minimum.

$$y(2) = \frac{1}{1}\left[0 + 2 \times 0.25 + \frac{2(2-1)}{2!} \times -0.50 + \frac{2(2-1)(2-2)}{3!} \times 3 + \cdots\right]$$

$$= 1\left[0 + 0.50 - 0.50 + 0\right]$$

$\boxed{y(2) = 0}$ minimum.

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UNIT V

1. BETA AND GAMMA FUNCTIONS
2. RELATION BETWEEN THEM.
3. EVALUATION OF MULTIPLE INTEGRALS USING BETA AND GAMMA FUNCTIONS.

I. Beta and Gamma functions

Def 1 (Beta fn).

$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m>0$, $n>0$ is known as Beta function and it is denoted by $B(m,n)$.

ie $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m,n>0$.

Def 2 (Gamma function)

$\int_0^\infty x^{n-1}e^{-x}\,dx$, for $n>0$, is known as Gamma function and it is denoted by $\Gamma(n)$.

ie $\int_0^\infty x^{n-1}e^{-x}\,dx$, for $n>0$.

II. Recurrence formula of Gamma functions.

We know that $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$, for $n>0$.

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx, \text{ for } n>-1$$

Integrating by parts, we know that $\int u\,dv = uv - \int v\,du$

taking $u = x^n$, $dv = e^{-x}dx$.

On differentiating u, we get $du = nx^{n-1}dx$.

On integrating $dv = e^{-x} dx$, we get

$$v = \frac{e^{-x}}{-1} = -e^{-x}$$

$$\therefore \int_0^\infty x^n e^{-x} dx = \left[x^n (-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x}) n x^{n-1} dx$$

$$= \left[-\infty^n e^{-\infty} + 0^n e^{-0}\right] + \int_0^\infty e^{-x} n x^{n-1} dx$$

$$= 0 + \int_0^\infty e^{-x} n x^{n-1} dx$$

$$\therefore \Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n \int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\,\Gamma(n)$$

$$\therefore \Gamma(n+1) = n\Gamma(n), \text{ if } n > -1.$$

<u>Corrolary</u>

1. $\Gamma(n+1) = n!$, when n is a positive integer.

<u>Proof:</u> W.K.T $\Gamma(n+1) = n\Gamma(n)$

$$= n\Gamma(n)$$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$\vdots$$

$$= n(n-1)(n-2)(n-3)\ldots\ldots 1\,\Gamma(1).$$

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$$\therefore \Gamma(1) = \int_0^\infty e^{-x} x^0 \, dx = \int_0^\infty e^{-x} \, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = \left[-e^{-x}\right]_0^\infty$$

$$= -e^{-\infty} - (-e^{-0})$$

$$= 0 + \frac{1}{e^0} = 0 + \frac{1}{1} \quad \left[\because e^{-\infty} = 0\right]$$

$$\therefore \Gamma(1) = 1.$$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)\cdots 2.1.$$

$$\boxed{\Gamma(n+1) = n!}$$

ii). **Properties of Beta functions**

$$B(m,n) = B(n,m).$$

Proof:

W.K.T, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx.$

put $x = 1-y$, $\Rightarrow dx = 0 - dy = -dy$.

Limits

when $x = 1$

$\Rightarrow 1 = 1-y \Rightarrow y = 0.$

when $x = 0$

$\Rightarrow 0 = 1-y \Rightarrow y = 1.$

now $B(m,n) = \int_1^0 (1-y)^{m-1} (1-(1-y))^{n-1} (-dy)$

$$= \int_1^0 (1-y)^{m-1} (1-1+y)^{n-1} (-dy)$$

$$= -\int_1^0 (1-y)^{m-1} y^{n-1} dy.$$

$$= -\int_1^0 y^{n-1} (1-y)^{m-1} dy.$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} dy. \quad \left[\because \int_0^1 f(x)dx = -\int_1^0 f(x)dx\right]$$

$$\boxed{B(m,n) = B(n,m).}$$

(2). $B(m,n)$ can be expressed as a definite integral with $0, \infty$ as limits.

<u>Proof:</u> By def $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$

put $x = \dfrac{y}{1+y}$

$$\Rightarrow dx = \frac{(1+y)dy - y(dy)}{(1+y)^2}$$

$$= \frac{dy + \cancel{y\,dy} - \cancel{y\,dy}}{(1+y)^2} = \frac{dy}{(1+y)^2}.$$

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$$dx = \frac{1}{(1+y)^2}\,dy.$$

Limits:

when $x = 0 \Rightarrow \dfrac{y}{1+y} = 0 \Rightarrow y = 0.$

when $x = 1 \Rightarrow \dfrac{y}{1+y} = 1 \Rightarrow 1+y = y$

$\Rightarrow 1 = 0$

$\therefore\ y = \infty.$

$$\text{Hence } B(m,n) = \int_0^{\infty} \left(\frac{y}{1+y}\right)^{m-1} \left(1-\frac{y}{1+y}\right)^{n-1} \frac{dy}{(1+y)^2}$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m-1}} \left(\frac{1+\cancel{y}-\cancel{y}}{1+y}\right)^{n-1} \frac{dy}{(1+y)^2}$$

$$= \int_0^{1} \frac{y^{m-1}}{(1+y)^{m-1}} \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}\,dy$$

$$B(m,n) = \int_0^{1} \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.$$

3. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$

~~Sol~~ Proof:

We know that

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \rightarrow (1)$$

put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta \, d\theta$

Limits

when $x = 0$

$$\Rightarrow 0 = \sin^2\theta \Rightarrow \theta = 0 \quad [\because \sin(0) = 0]$$

when $x = 1$

$$1 = \sin^2\theta \Rightarrow \theta = \pi/2 \quad [\because \sin\pi/2 = 1.]$$

Now,

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \, (\cos^2\theta)^{n-1} 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-2}\theta \cos\theta \, d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Relation between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Proof:

We know that $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$.

put $x = xz$, where z is a positive constant with respect to x.

$$x = xz \Rightarrow dx = z\,dx. \quad \text{no change in Limits}$$

$$\therefore \Gamma(n) = \int_0^\infty e^{-xz} (xz)^{n-1} z\,dx$$

$$= \int_0^\infty e^{-xz} x^{n-1} z^n\,dx$$

$$\Gamma(n) = \int_0^\infty e^{-xz} x^{n-1} z^n\,dx$$

multiply $e^{-z} z^{m-1}$ on both sides, we get.

$$\Gamma(n)\,e^{-z} z^{m-1} = \int_0^\infty e^{-xz} x^{n-1} z^n\,dx \cdot e^{-z} z^{m-1}$$

Integrating on both sides, with the limits 0 to ∞.

$$\Gamma(n) \int_0^\infty e^{-z} z^{m-1}\,dz = \int_0^\infty \int_0^\infty e^{-xz-z} x^{n-1} z^{n+m-1}\,dx\,dz$$

$$= \int_0^\infty \left[\int_0^\infty e^{-z(x+1)} z^{m+n-1}\,dz \right] x^{n-1}\,dx \rightarrow (1)$$

put $z(1+x) = y$

$$\Rightarrow z = \frac{y}{1+x} \Rightarrow dz = \frac{(1+x)\,dy - y(0)}{(1+x)^2}$$

$$\therefore dz = \frac{(1+x)\,dy}{(1+x)^{\cancel{2}}} = \frac{dy}{(1+x)}$$

Limits:

when $z = 0$

$y = 0$

When $z=\infty$, $y = \infty$.

Therefore, now $\int_0^\infty e^{-z(x+1)} z^{m+n-1}\,dz$ is

$$= \int_0^\infty e^{-y} \left(\frac{y}{1+x}\right)^{m+n-1} \frac{dy}{(1+x)}$$

$$= \int_0^\infty e^{-y} \frac{y^{m+n-1}}{(1+x)^{m+n-1}} \cdot \frac{1}{(1+x)} \cdot dy .$$

$$= \frac{1}{(1+x)^{m+n}} \int_0^\infty e^{-y} y^{m+n-1}\, dy$$

$$\therefore \int_0^\infty e^{-z(x+1)} z^{m+n-1}\,dz = \frac{1}{(1+x)^{m+n}} \Gamma(m+n) \rightarrow (2).$$

[$\because$ def of Gamma]

take (2) in (1), we get

$$\Gamma(n) \int_0^\infty e^{-z} z^{m-1}\,dz = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} \cdot x^{n-1}\,dx.$$

$$\Gamma(n)\,\Gamma(m) = \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$= \Gamma(m+n)\,\beta(m,n)$$, by property (2) of Beta fun

$$\frac{\Gamma(n)\,\Gamma(m)}{\Gamma(m+n)} = \beta(m,n).$$ Hence proved.

Corollary 1

Prove that $\Gamma(1/2) = \sqrt{\pi}$.

Proof:

We know that $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

put $m = n = 1/2$

$$\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = B(1/2, 1/2) \rightarrow (1)$$

We know that $B(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}x \; \cos^{2n-1}x \, dx,$

$$B(1/2, 1/2) = 2\int_0^{\pi/2} \sin^{2(1/2)-1}x \cdot \cos^{2(1/2)-1}x \, dx.$$

$$= 2\int_0^{\pi/2} \sin^0 x \; \cos^0 x \, dx$$

$$= 2\int_0^{\pi/2} dx \qquad \left[\because \sin^0 x = \cos^0 x = 1\right]$$

$$= 2\left[x\right]_0^{\pi/2}$$

$$= 2\left(\pi/2 - 0\right)$$

$$B(1/2, 1/2) = \pi \rightarrow (2).$$

Take (2) in (1)

$$\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = \pi$$

$$\Rightarrow \left(\Gamma(1/2)\right)^2 = \pi \qquad \left[\because \Gamma(1) = 1\right]$$

Square root on both sides, we get.

$\Gamma(1/2) = \sqrt{\pi}$. Hence proved.

— x —

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Corrolary (2).

Prove that $\Gamma(1/4)\,\Gamma(3/4) = \sqrt{2}\cdot\pi$.

Proof:

We know that $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

$$\therefore \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2} B(m,n)$$

put $2m = p$ and $2n = q$

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta\, d\theta = \frac{1}{2} B(p/2, q/2)$$

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta\, d\theta = \frac{1}{2} \frac{\Gamma(p/2)\,\Gamma(q/2)}{\Gamma\left(\frac{p+q}{2}\right)} \rightarrow (1)$$

[$\because$ Relation between Beta & Gamma function]

If we put $q = 1$ in (1) we get

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{1-1}\theta\, d\theta = \frac{1}{2} \frac{\Gamma(p/2)\,\Gamma(1/2)}{\Gamma\frac{(p+1)}{2}}$$

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{0}\theta\, d\theta = \frac{1}{2} \frac{\Gamma(p/2)\,\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)}$$

$$\int_0^{\pi/2} \sin^{p-1}\theta\, d\theta = \frac{1}{2} \frac{\Gamma(p/2)\,\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)} \rightarrow (2)$$

If we put $p = q$ in (1), we get.

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{p-1}\theta\, d\theta = \frac{1}{2} \frac{\left(\Gamma(p/2)\right)^2}{\Gamma p}$$

$$\int_0^{\pi/2} \frac{2^{p-1}\sin^{p-1}\theta \cos^{p-1}\theta}{2^{p-1}}\, d\theta = \frac{1}{2} \frac{\left(\Gamma(p/2)\right)^2}{\Gamma(p)}$$

$$\frac{1}{2^{p-1}}\int_0^{\pi/2} \sin^{p-1} 2\theta \, d\theta = \frac{1}{2}\frac{\left(\Gamma(p/2)\right)^2}{\Gamma(p)} \quad \left[\because \sin 2\theta = 2\sin\theta\cos\theta\right]$$

put $2\theta = \varphi \Rightarrow 2\,d\theta = d\varphi \Rightarrow d\theta = \frac{d\varphi}{2}$

Limits:
when $\theta = 0 \Rightarrow 2(0) = \varphi \Rightarrow \varphi = 0.$
when $\theta = \pi/2 \Rightarrow \varphi = \cancel{2}(\pi/\cancel{2}) = \pi \Rightarrow \varphi = \pi.$

$$\therefore \frac{1}{2^{p-1}}\int_0^{\pi} \sin^{p-1}\varphi \, \frac{d\varphi}{\cancel{2}} = \frac{\frac{1}{\cancel{2}}\left(\Gamma(p/2)\right)^2}{\Gamma(p)}$$

$$\frac{2}{2^{p-1}}\int_0^{\pi/2} \sin^{p-1}\varphi \, d\varphi = \frac{\left(\Gamma(p/2)\right)^2}{\Gamma(p)} \quad \text{—— } (A)$$

Take (2) in (A)

$$\frac{\cancel{2}}{2^{p-1}} \cdot \frac{1}{\cancel{2}} \frac{\cancel{\Gamma(p/2)}\,\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)} = \frac{\left(\Gamma(p/2)\right)^{\cancel{2}}}{\Gamma(p)}$$

$$\frac{1}{2^{p-1}} \cdot \frac{\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)} = \frac{\Gamma(p/2)}{\Gamma(p)}$$

$$\Rightarrow \frac{1}{2^{p-1}}\Gamma(1/2)\,\Gamma(p) = \Gamma(p/2)\,\Gamma\left(\frac{p+1}{2}\right)$$

whence $\frac{1}{2^{p-1}}\sqrt{\pi}\,\Gamma(p) = \Gamma(p/2)\cdot\Gamma\left(\frac{p+1}{2}\right) \rightarrow (3) \quad \left[\because \Gamma(1/2) = \sqrt{\pi}\right]$

putting $p = 2n$ in (3), we get

$$\frac{1}{2^{2n-1}}\sqrt{\pi}\,\Gamma(2n) = \Gamma\left(\frac{2n}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)$$

$$= \Gamma(n)\,\Gamma\left(\frac{2n}{2}+\frac{1}{2}\right)$$

$$\Gamma(n)\,\Gamma(n+1/2) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \rightarrow (4)$$

put $n = 1/4$ in (4), we get

$$\Gamma(1/4)\,\Gamma(1/4+1/2) = \frac{\sqrt{\pi}\,\Gamma 2(1/4)}{2^{2(1/4)-1}}$$

$$\Gamma(1/4)\,\Gamma(3/4) = \frac{\sqrt{\pi}\,\Gamma(1/2)}{2^{1/2-1}} = \frac{\sqrt{\pi}\,\Gamma(1/2)}{2^{-1/2}}$$

$$\Gamma(1/4)\,\Gamma(3/4) = \frac{\sqrt{\pi}\,\Gamma(1/2)}{\frac{1}{2^{1/2}}}$$

$$= \sqrt{\pi}\,\Gamma(1/2)\,2^{1/2}$$

$$\Gamma(1/4)\,\Gamma(3/4) = \sqrt{\pi}\,\sqrt{\pi}\,2^{1/2}$$

$$= \pi \cdot \sqrt{2}.$$

Hence $\Gamma(1/4) \cdot \Gamma(3/4) = \sqrt{2} \cdot \pi.$

Evaluate: $\int_0^\infty e^{-x^2}\,dx.$

(1)

Soln. put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{dt}{2x}$

$\therefore x = \sqrt{t} \Rightarrow dx = \frac{dt}{2\sqrt{t}}.$

Now, $\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-t} \cdot \frac{dt}{2\sqrt{t}}$

$$= \frac{1}{2}\int_0^\infty e^{-t}\frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}\,dt$$

$$= \frac{1}{2}\,\Gamma(1/2)$$

$$\therefore \int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

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(2) Evaluate $\int_0^1 x^m (\log 1/x)^n \, dx$.

Soln:

Given $\int_0^1 x^m (\log 1/x)^n \, dx \rightarrow (1)$

put $\log 1/x = t \Rightarrow \frac{1}{x} = e^{t} \Rightarrow x = e^{-t}$

$\therefore dx = e^{-t}(-1)dt = -e^{-t}dt$.

Limit:

when $x = 0$, $\log \frac{1}{0} = \infty = t$. $[\because \log \infty = \infty]$

when $x = 1$, $\log 1/1 = t = 0$ $[\because \log 1 = 0]$

Now, the

$$\int_0^1 x^m (\log 1/x)^n \, dx = \int_\infty^0 (e^{-t})^m t^n . -e^{-t} dt.$$

$$= -\int_\infty^0 e^{-tm} t^n e^{-t} dt$$

$$= +\int_0^\infty e^{-tm-t} t^n dt$$

$$= \int_0^\infty e^{-t(m+1)} t^n dt.$$

put $(m+1)t = y \Rightarrow t = \frac{y}{m+1} \Rightarrow dt = \frac{1}{m+1} dy$.

Limit when $t = 0$, $y = 0$

when $t = \infty$ $y = \infty$

Now, $\int_0^\infty e^{-t(m+1)} t^n dt = \int_0^\infty e^{-y} \left(\frac{y}{m+1}\right)^n dt \frac{1}{m+1}$

$$= \int_0^\infty e^{-y} \frac{y^n}{(m+1)^n} \cdot \frac{1}{(m+1)} \cdot dt$$

$$= \int_0^\infty e^{-y}\, y^n \cdot \frac{1}{(m+1)^{n+1}}\, dy .$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y}\, y^n\, dy .$$

$$\therefore \int_0^1 x^m (\log 1/x)^n dx = \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$$

③. Evaluate (i) $\int_0^1 x^7 (1-x)^8 dx$ (ii) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$.

Soln (i) $\int_0^1 x^7 (1-x)^8 dx = \beta(7+1, 8+1)$ by def

$$= \frac{\Gamma(8)\,\Gamma(9)}{\Gamma(8+9)} = \frac{7!\ 8!}{\Gamma(17)} \quad \left[\because \Gamma(n+1) = n!\right]$$

30 × 11 = 330

$$= \frac{7!\,.\,8!}{16!}$$

$$= \frac{\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\ .\ \cancel{8!}}{16\times15\times\cancel{14}\times13\times\cancel{12}\times11\times\cancel{10}\times\cancel{9}\times\cancel{8!}}$$

$$= \frac{1}{16\times15\times2\times13\times11} = \frac{1}{330\times16\times13}$$

(ii) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta . = \frac{1}{2}\beta\left(\frac{7+1}{2}, \frac{5+1}{2}\right)$ $\quad \because \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

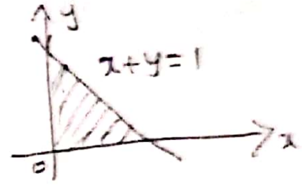
$$= \frac{1}{2}\beta(4,3) = \frac{1}{2}\frac{\Gamma(4)\,\Gamma(3)}{\Gamma(7)}$$

$$= \frac{1}{2}\cdot\frac{3!\ 2!}{6!} = \frac{1}{2}\times\frac{\cancel{3!}\times\cancel{2}}{6\times5\times\cancel{4}\times\cancel{3!}} = \frac{1}{120} \ //.$$

H.W.

Evaluation of multiple Integrals using Beta & Gamma fns

① Evaluate the integral $\iint x^p y^q \, dy \, dx$ over the $x > 0, y > 0, x + y \le 1$ in terms of Ga



Soln The region of the triangle is

$$\iint x^p y^q \, dy \, dx = \int_0^1 \int_0^{1-x} x^p y^q \, dy \, dx.$$

$\because x = 0 \Rightarrow y = 1 - x = 1 - 0 = 1.$

$x = 0 \Rightarrow y = 1 - x.$

$$= \int_0^1 x^p \int_0^{1-x} y^q \, dy \, dx$$

$$= \int_0^1 x^p \left[\frac{y^{q+1}}{q+1} \right]_0^{1-x} dx = \int_0^1 x^p \left[\frac{(1-x)^{q+1}}{q+1} - 0 \right] dx$$

$$= \int_0^1 \frac{x^p (1-x)^{q+1}}{q+1} \, dx = \frac{1}{q+1} \int_0^1 x^p (1-x)^{q+1} \, dx.$$

$$= \frac{1}{q+1} \beta(p+1, q+2) \qquad [\because \text{by def}]$$

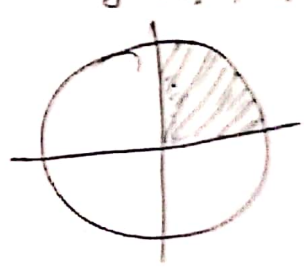
$$= \frac{1}{q+1} \frac{\Gamma(p+1)\, \Gamma(q+2)}{\Gamma(p+1+q+2)} = \frac{1}{\cancel{q+1}} \cdot \frac{\Gamma(p+1)\, \cancel{q+1}\, \Gamma(q+1)}{\Gamma(p+q+3)}$$

$$\therefore \iint x^p y^q \, dy \, dx = \frac{\Gamma(p+1)\, \Gamma(q+1)}{\Gamma(p+q+3)}$$

② Evaluate the integral $\iint x^p y^q \, dx \, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ in terms of Gamma functions. Deduce (i) the Area of the circle and (ii) the co-ordinates of the centroid of a quadrant of the circle.

Soln The positive quadrant of the circle is given by $x \ge 0, y \ge 0$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 \le 1.$$



put $\frac{x}{a} = X^{1/2}$, $\frac{y}{a} = Y^{1/2}$

$x = a X^{1/2}$, $y = a Y^{1/2}$

$$dx = a \cdot \tfrac{1}{2} x^{1/2 - 1} dx \quad , \quad dy = a \cdot \tfrac{1}{2} y^{1/2 - 1} dy$$

$$dx = a \tfrac{1}{2} x^{-1/2} dx \quad , \quad dy = a \cdot \tfrac{1}{2} y^{-1/2} dy$$

$$\therefore \iint x^p y^q \, dx\, dy = \int_0^1 \int_0^{1-y} (a x^{1/2})^p (a y^{1/2})^q \frac{a}{2} x^{-1/2} \frac{a}{2} y^{-1/2} dx\, dy$$

$$[\because x>0, y>0 \; x+y \leq 1]$$

$$= \int_0^1 \int_0^{1-y} a^p x^{p/2} a^q y^{q/2} \frac{a}{2} x^{-1/2} \frac{a}{2} y^{-1/2} dx\, dy$$

$$= a^p a^q \frac{a^2}{4} \int_0^1 \int_0^{1-y} x^{p/2 - 1/2} y^{q/2 - 1/2} dx\, dy.$$

$$= \frac{a^{p+q+2}}{4} \int_0^1 \int_0^{1-y} x^{\frac{p-1}{2}} y^{\frac{q-1}{2}} dx\, dy$$

$$= \frac{a^{p+q+2}}{4} \int_0^1 y^{\frac{q-1}{2}} \left[\frac{x^{\frac{p-1}{2}+1}}{\frac{p-1}{2}+1} \right]_0^{1-y} dy$$

$$= \frac{a^{p+q+2}}{4} \int_0^1 y^{\frac{q-1}{2}} \left[\frac{(1-y)^{\frac{p+1}{2}}}{\frac{p+1}{2}} - 0 \right] dy$$

$$= \frac{a^{p+q+2}}{4} \int_0^1 y^{\frac{q-1}{2}} \cdot \frac{2 \cdot (1-y)^{\frac{p+1}{2}}}{p+1} dy$$

$$= \frac{a^{p+q+2}}{\cancel{4}_2} \cdot \frac{\cancel{2}}{p+1} \int_0^1 y^{\frac{q-1}{2}} \cdot (1-y)^{\frac{p+1}{2}} dy$$

$$= \frac{a^{p+q+2}}{2(p+1)} \beta\left(\frac{q-1}{2}+1, \frac{p+1}{2}+1\right)$$

$$= \frac{a^{p+q+2}}{2(p+1)} \beta\left(\frac{q+1}{2}, \frac{p+1}{2}+1\right)$$

$$= \frac{a^{p+q+2}}{2(p+1)} \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{p+1}{2}+1\right)}{\Gamma\left\{\frac{q+1}{2}+\frac{p+1}{2}+1\right\}}$$

$$= \frac{a^{p+q+2}}{2(p+1)} \cdot \frac{\Gamma\left(\frac{q+1}{2}\right)\frac{p+1}{2}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{q+p+2+2}{2}\right)}$$

$$= \frac{a^{p+q+2}}{2.(p+1)} \cdot \frac{\Gamma\left(\frac{q+1}{2}\right)\frac{p+1}{2}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+q+4}{2}\right)}$$

$$= \frac{a^{p+q+2}}{2.2} \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+q}{2}+2\right)}$$

$$= \frac{a^{p+q+2}}{4} \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+q}{2}+2\right)}$$

(i) Area of the circle is $4\iint dx\,dy$ over the region $x \geq 0$, $y \geq 0$, $x^2+y^2 \leq a^2$.

In this case $p=0$, $q=0$

$$\therefore \text{Area of the circle} = \cancel{4} . \frac{a^{0+0+2}}{\cancel{4}} . \frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{0+0}{2}+2\right)}$$

$$= \frac{a^2\,\Gamma(1/2)\,\Gamma(1/2)}{\Gamma 2} = \frac{a^2\,\Gamma(1/2)\,\Gamma(1/2)}{\Gamma 1+1}$$

$$\therefore \text{Area of the circle} = \frac{a^2\pi}{1\Gamma 1} = \frac{a^2\pi}{1} = a^2\pi.$$

(ii) Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centroid of the quadrant of the circle.

$$\bar{x} = \frac{\iint x\,dy\,dx}{\iint dy\,dx}$$ both the integrals being taken over the region $x \geq 0$, $y \geq 0$, $x^2+y^2 \leq a^2$.

In the integral $\iint x^p y^q \, dx\, dy$ if we $p=1, q=0$, we get the numerator.

$\therefore$ Numerator $= \iint x \, dy \, dx \qquad , \quad p=1, q=0.$

$$= \frac{a^{1+2}}{4} \cdot \frac{\Gamma\frac{1+1}{2} \; \Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{1+0}{2}+2\right)}$$

$$= \frac{a^3}{4} \cdot \frac{\Gamma 1 \; \Gamma \frac{1}{2}}{\Gamma \frac{5}{2}} = \frac{a^3 \cdot 1 \cdot \sqrt{\pi}}{4 \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2}}$$

$$= \frac{a^3}{4} \cdot \frac{\sqrt{\pi}}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} = \frac{a^3}{4} \cdot \frac{1}{\frac{3}{4}}$$

$$= \frac{a^3}{4} \cdot \frac{4}{3} = \frac{a^3}{3}.$$

$$\therefore \bar{x} = \frac{\iint x \, dx\, dy}{\iint dx\, dy} = \frac{\frac{a^3}{3}}{\frac{1}{4} \cdot \pi a^2} \qquad \left[\because 4 \iint dx\, dy = \pi a^2 \right]$$

$$= \frac{a^{\cancel{3}}}{3} \cdot \frac{4}{\pi \cancel{a^2}}$$

$$\bar{x} = \frac{a}{3} \cdot \frac{4}{\pi}.$$

Illly $\bar{y} = \frac{4a}{3\pi}$

Hence the centroid is $(\bar{x}, \bar{y}) = \left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$
